Time - Energy Uncertainty

\[ \Delta t \cdot \Delta E \geq \frac{\hbar}{2} \]

Time for the system to evolve. If it never evolves \( \rightarrow \) constant \( E \).

Time Dependent Schrödinger Eq.

\[ i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V \psi \]

Separation of Variables (See Griffiths 2.1)

\[ \psi(x,t) = \phi(x) \varphi(t) \]

\[ \frac{\partial \varphi}{\partial t} = \psi \frac{\partial \varphi}{\partial \psi} \quad \Rightarrow \quad \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial x^2} \phi \]

\[ i \hbar \psi \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial x^2} \phi + V \psi \varphi \]

\[ i \hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial x^2} + V \]

constant \( \rightarrow \) constant (call \( i \hbar \) \( \hat{E} \))

\[ i \hbar \frac{\partial \phi}{\partial t} \]

\[ \frac{\partial \phi}{\partial t} = \frac{-i \hbar \hat{E}}{\hbar} \]

\[ \phi(t) = e^{-i \hat{E}/\hbar} \]

Time independent phase

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial x^2} + V \varphi = \hat{E} \varphi \]

\[ \left( \frac{\hbar^2}{2m} + V \right) \varphi = \hat{E} \varphi \]

Time Independent Schrödinger Eq.

Hamiltonian operator (total energy)
Stationary States

$$|\psi(x,t)|^2 = \int \psi^* e^{iE\tau/\hbar} \psi e^{-iE\tau/\hbar} \ dx = |\mathcal{V}(x)|^2$$

$$\sigma_H^2 = \langle H^2 \rangle - \langle H \rangle^2 = \hbar^2 - E^2 = 0$$

General Solution

$$\psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

Example potential: Infinite Square Well (See Griffiths 2.2)

Outside well: $\psi(x) = 0$ to cancel $V = \infty$ and allow normalization

Inside well: $V = 0 \rightarrow \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = E \psi$

Let $K = \sqrt{\frac{2mE}{\hbar}} \frac{\partial^2 \psi}{\partial x^2} = -k^2 \psi$

Classical harmonic oscillator! Hookes Law!
Solutions: $\psi(x) = A \sin kx + B \cos kx$
Boundary conditions (and continuity)

at \( x = 0 \): \( \psi(0) = 0 = A \sin \theta + B \cos \theta \Rightarrow B = 0 \)

\[ \therefore \psi(x) = A \sin kx \]

at \( x = a \): \( \psi(a) = 0 = A \sin ka \Rightarrow ka = 0 \pm n \pi \]  

negative \( ka \) are repeat solutions and \( ka = 0 \) is not normalizable.

\[ k_n = \frac{n \pi}{a} : n = 1, 2, 3, \ldots \]

\[ E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \]  

Normalize to get \( A = \sqrt{\frac{2}{a}} \)

\[ \psi_n(x) = \sqrt{\frac{2}{a}} \sin \left( \frac{n \pi}{a} x \right) \]

\[ n = 1 \quad n = 2 \quad n = 3 \quad n = 4 \]

This is the expected position, but in each case there is zero probability of finding the particle there!

The discrete states are a result of the particle being bound in a potential. At last we have seen something quantized about quantum mechanics.