HW 7 Solution

1.14. (a). Normalization condition:

$$\int_{-\infty}^{\infty} dx\, \psi^* \psi = 1$$

$$\Rightarrow \quad A^2 \int_{-\infty}^{\infty} dx\, e^{-\frac{2am}{\hbar}x^2} = 1$$

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{2am}{\hbar}x^2} = 2\int_{0}^{\infty} dx\, e^{-\frac{2am}{\hbar}x^2}$$

$$= \sqrt{\frac{\pi\hbar}{2am}}$$

$$\Rightarrow \quad A^2 = \left(\frac{\pi\hbar}{2ma}\right)^{-\frac{1}{2}}$$

$$A = \left(\frac{2ma}{\pi\hbar}\right)^{\frac{1}{4}}$$

(b) In Schrodinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V(x)\psi$$

$$\Rightarrow \quad V(x)\psi = \left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial^2 x}\right)\psi$$

In this problem:

$$i\hbar\frac{\partial}{\partial t}\psi = \hbar a\,\psi$$

$$\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = \left(-\hbar a + 2ma^2x^2\right)\psi$$

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial^2 x}\right)\psi = 2ma^2x^2\psi$$

$$\Rightarrow \quad V(x) = 2ma^2x^2$$

(c). $\langle x\rangle = \int_{-\infty}^{\infty} \psi^* x\, \psi\, dx = A^2\int_{-\infty}^{\infty} x\, e^{-\frac{2ma}{\hbar}x^2} dx = 0$ (odd function)

$$\langle x^2\rangle = \int_{-\infty}^{\infty} \psi^* x^2 \psi\, dx = A^2\int_{-\infty}^{\infty} x^2 e^{-\frac{2ma}{\hbar}x^2} dx = A^2 \cdot \frac{1}{2}\sqrt{\frac{\pi\hbar^3}{8m^3a^3}}$$

$$= \frac{\hbar}{4am}$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi \, dx = A^2 \int_{-\infty}^{\infty} (-i\hbar)\left(\frac{2mq}{\hbar^2}\right) x \, e^{-\frac{2mq}{\hbar}x^2} dx = 0$$

(odd function)

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi^* \left(-\hbar^2 \frac{\partial^2}{\partial x^2}\right) \psi \, dx = A^2(-\hbar^2) \int_{-\infty}^{\infty} \left(-\frac{2mq}{\hbar} e^{-\frac{2am}{\hbar}x^2} + \frac{4a^2m^2}{\hbar^2} x^2 e^{-\frac{2mq}{\hbar}x^2}\right) dx$$

$$= A^2(-\hbar^2)\left(-\sqrt{\frac{2ma\pi}{\hbar}} + \frac{1}{2}\sqrt{\frac{2ma\pi}{\hbar}}\right)$$

$$= \hbar A^2 \frac{1}{2}\sqrt{2ma\pi\hbar}$$

$$= ma\hbar$$

(d). $$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{4ma}}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{ma\hbar}$$

$$\sigma_x \cdot \sigma_p = \frac{\hbar}{2}$$

This product is consistant with uncertainty principle ($\sigma_x \cdot \sigma_p \geq \frac{\hbar}{2}$)

2.2 Schrodinger Equation:

$$\frac{d^2}{dx^2}\psi(x) = \frac{2m}{\hbar}[V(x) - E]\psi(x)$$

Now we assume $V(x) = V_{min}$ (constant) for simplicity. if $E < V_{min}$ then we set $k^2 = \frac{2m}{\hbar}(V_{min} - E) > 0$.

$$\frac{d^2}{dx^2}\psi(x) = k^2\psi(x)$$

$$\Rightarrow \quad \psi(x) = Ae^{kx} + Be^{-kx}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \psi^*(x)\psi(x)\,dx \to \infty$$

This means we can not have normalized wavefunction, which is not ~~a~~normalizable.

On the Contrary, E must be greater than V_{min} to get a normalizable wavefunction.

classical analog:

In classical mechanics:

$$E_{tot} = E_p + E_k$$

(E_{tot}: total energy; E_p: potential energy; E_k: kinetic energy)

kinetic energy should always be non-negative:

$$E_k = \frac{1}{2}mv^2 \geq 0$$

This means:

$$E_{tot} \geq E_{p\,min}$$

In order to have a physical motion, E_{tot} should be greater than (or equal to) the minimum of the potential energy.

2.3. If $E=0$, than for infinite well

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) = 0$$

$$\Rightarrow \psi(x) = Ax + B$$

boundary condition:

$$\psi(0) = \psi(a) = 0$$

$$\Rightarrow A = B = 0$$

$$\Rightarrow \psi(x) = 0 \quad \text{(trivial solution)}.$$

If $E<0$, than

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) = E\psi(x)$$

$$\frac{\partial^2}{\partial x^2}\psi(x) = k^2\psi(x) \quad \left(k^2 = \frac{-2mE}{\hbar^2} > 0\right)$$

$$\Rightarrow \psi(x) = Ae^{kx} + Be^{-kx}$$

boundary condition:

$$\psi(0) = \psi(a) = 0$$

$$\Rightarrow \quad A = B = 0$$

$$\Rightarrow \quad \psi(x) = 0 \quad \text{(trivial solution)}$$

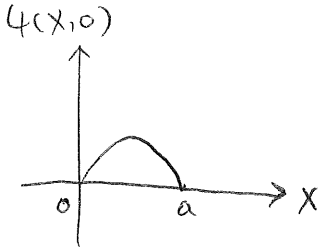
So if $E \leq 0$, we can only get trivial solution, which has no physical meaning.

2.8. (a). Normalization:

$$A^2 \int_0^a dx\, x^2 (a-x)^2 = 1$$

$$\Rightarrow \quad A^2 = \frac{30}{a^5}$$

$$\Rightarrow \quad A = \sqrt{\frac{30}{a^5}}$$



This has no node (except two ends), which looks like $n=$ [redacted] state (ground state).

Since: $E_n = \frac{\pi^2 \hbar^2 n^2}{2ma^2}$

expected energy: $\bar{E}_1 = \frac{\pi^2 \hbar^2}{2ma^2}$

(b). $$\langle x \rangle = \int_0^a \psi^* x \psi\, dx = A^2 \int_0^a x^3 (a-x)^2 dx = \frac{a}{2}$$

$$\langle p \rangle = \int_0^a \psi^* (-i\hbar) \frac{\partial}{\partial x} \psi\, dx = -i\hbar A^2 \int_0^a x(a-x)(a-2x)\, dx$$

$$\overset{y=\frac{a}{2}-x}{=} (-i\hbar 2A^2) \int_{-\frac{a}{2}}^{\frac{a}{2}} y \underbrace{\left(y^2 - \left(\frac{a}{2}\right)^2\right)}_{\text{odd function.}} dy$$

$$= 0$$

$$\langle H \rangle = \int_0^a \psi^* \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\right) \psi\, dx = -\frac{\hbar^2 A^2}{2m} \int_0^a x(a-x)(-2)\, dx$$

$$= \frac{5\hbar^2}{ma^2}$$

Because $\frac{\pi^2}{2} \approx 5 \Rightarrow E_1 \approx \langle H \rangle$ (E_1 is rather good estimation)