

Calculate the partition function $Z = \sum_i e^{-E_i/kT}$ for the following systems with energy spectrum E_i .

- (a) Two-level system with energies $E_1 = 0$ and $E_2 = \Delta$

$$\begin{aligned} Z &= e^{-\frac{E_1}{kT}} + e^{-\frac{E_2}{kT}} \\ &= 1 + e^{-\frac{\Delta}{kT}} \\ &= 2e^{-\Delta/2kT} \cosh(\Delta/2kT) \end{aligned}$$

- (b) Harmonic oscillator with energy levels $E_n = \hbar\omega \left(n + \frac{1}{2}\right)$, $n=0,1,2\dots$

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} e^{-\frac{\hbar\omega}{kT} \left(n + \frac{1}{2}\right)} \\ &= e^{-\frac{\gamma}{2}} \sum_{n=0}^{\infty} e^{-n\gamma} \quad \text{where } \gamma = \frac{\hbar\omega}{kT} \\ &= \frac{e^{-\frac{\gamma}{2}}}{1 - e^{-\gamma}} = \frac{1}{e^{\gamma/2} - e^{-\gamma/2}} = \frac{1}{2} \operatorname{csch}\left(\frac{\hbar\omega}{kT}\right) \end{aligned}$$

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