

AN INTRODUCTION TO MECHANICS

Daniel Kleppner and Robert Kolenkow

SECOND EDITION

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Robert Kolenkow

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12 THE SPECIAL THEORY OF RELATIVITY

12.1	Introduction	440
12.2	The Possibility of Flaws in Newtonian Physics	440
12.3	The Michelson–Morley Experiment	442
12.4	The Special Theory of Relativity	445
	12.4.1 Synchronizing Clocks	445
	12.4.2 The Principle of Relativity	446
12.5	Transformations	447
	12.5.1 The Galilean Transformation	448
12.6	Simultaneity and the Order of Events	450
12.7	The Lorentz Transformation	451
	12.7.1 Michelson–Morley Revisited	454
12.8	Relativistic Kinematics	454
	12.8.1 Time Dilation	455
	12.8.2 Length Contraction	457
	12.8.3 Proper Time and Proper Length	459
	12.8.4 Are Relativistic Effects Real?	459
12.9	The Relativistic Addition of Velocities	463
12.10	The Doppler Effect	466
	12.10.1 The Doppler Effect in Sound	466
	12.10.2 The Relativistic Doppler Effect	467
	12.10.3 The Doppler Effect Off the Line of Motion	468
12.11	The Twin Paradox	470
	<i>Problems</i>	472

12.1 Introduction

In the centuries following publication of the *Principia*, Newtonian dynamics was accepted whole-heartedly not only because of its enormous success in explaining planetary motion but also in accounting for all motions commonly encountered on the Earth. Physicists and mathematicians (often the same people) created elegant reformulations of Newtonian physics and introduced more powerful analytical and calculational techniques, but the foundations of Newtonian physics were assumed to be unassailable. Then, on June 30 1905, Albert Einstein presented his special theory of relativity in his publication *The Electrodynamics of Moving Bodies*. The English translation, available on the web, is reprinted from *Relativity: The Special and General Theory*, Albert Einstein, Methuen, London (1920). The original publication is *Zur Elektrodynamik bewegter Körper*, *Annalen der Physik* 17 (1905). Einstein's paper transformed our fundamental view of space, time, and measurement.

The reason that Newtonian dynamics went unchallenged for over two centuries is that although we now realize that it is only an approximation to the laws of motion, the approximation is superb for motion with speed much less than the speed of light, $c \approx 3 \times 10^8$ m/s. Relativistic modifications to observations of a body moving with speed v typically involve a factor of v^2/c^2 . Most familiar phenomena involve speeds $v \ll c$. Even for the high speed of an Earth-orbiting satellite, $v^2/c^2 \approx 10^{-10}$. There is one obvious exception to this generalization about speed: light itself. Thus, it is hardly surprising that the problems that triggered Einstein's thinking concerned not mechanics but light, problems that grew out of Einstein's early fascination with Maxwell's electromagnetic theory—the theory of light.

12.2 The Possibility of Flaws in Newtonian Physics

The German physicist and philosopher Ernst Mach first pointed out the possibility of flaws in Newtonian thought. Although Mach proposed no changes to Newtonian dynamics, his analysis impressed the young Einstein and was crucial in the revolution shortly to come. Mach's 1883 text *The Science of Mechanics* incorporated the first incisive critique of Newton's ideas about dynamics. Mach carefully analyzed Newton's explanation of the dynamical laws, taking care to distinguish between definitions, derived results, and statements of physical law. Mach's approach is now widely accepted; our discussion of Newton's laws in Chapter 2 is very much in Mach's spirit.

The Science of Mechanics raised the question of the distinction between absolute and relative motion. According to Mach, the fundamental weakness in Newtonian dynamics was Newton's conception of space and time. Newton avowed that he would forgo abstract speculation ("I do not make hypotheses") and deal only with observable facts, but he was not totally faithful to this resolve. In particular, consider the following

description of time that appears in the *Principia*. (The excerpt is condensed.) *Absolute, true and mathematical time, of itself and by its own true nature, flows uniformly on, without regard to anything external. Relative, apparent and common time is some sensible and external measure of absolute time estimated by the motions of bodies, whether accurate or inequable, and is commonly employed in place of true time; as an hour, a day, a month, a year.*

Mach commented “it would appear as though Newton in the remarks cited here still stood under the influence of medieval philosophy, as though he had grown unfaithful to his resolve to investigate only actual facts.” Mach went on to point out that since time is necessarily measured by the repetitive motion of some physical system, for instance the pendulum of a clock or the revolution of the Earth about the Sun, then the properties of time must be connected with the laws that describe the motions of physical systems. Simply put, Newton’s idea of time without clocks is metaphysical; to understand the properties of time we must observe the properties of clocks. As a prescient question, we might inquire whether a time interval observed on a moving clock has the same value as the interval observed on a clock at rest. A simple question? Yes indeed, except that the idea of absolute time is so natural that the eventual consequences of Mach’s critique, the relativistic description of time, still comes as something of a shock to students of science.

There are similar weaknesses in the Newtonian view of space. Mach argued that since position in space is determined using measuring rods, the properties of space can be understood only by investigating the properties of meter sticks. For example, does the length of a meter stick observed while it is moving agree with the length of the same meter stick at rest? To understand space we must look to nature, not to Platonic ideals.

Mach’s special contribution was to examine the most elemental aspects of Newtonian thought, to look critically at matters that might seem too simple to discuss, and to insist that correctly understanding nature means turning to experience rather than invoking mental abstractions. From this point of view, Newton’s assumptions about space and time must be regarded merely as postulates. Newtonian mechanics follows from these postulates, but other assumptions are possible and from them different laws of dynamics could follow.

Mach’s critique had no immediate effect but its influence was eventually profound. The young Einstein, while a student at the Polytechnic Institute in Zurich in the period 1897–1900, was much attracted by Mach’s work and by Mach’s insistence that physical concepts be defined in terms of observables. However, the most urgent reason for superseding Newtonian physics was not Mach’s critique but Einstein’s recognition that there were inconsistencies in interpreting the results of Maxwell’s electromagnetic theory, notwithstanding that Maxwell’s theory was considered the crowning achievement of classical physics.

The crucial event that triggered the theory of special relativity and decisively altered physics is generally taken to be the Michelson–Morley experiment, though it is not clear precisely what role this experiment actually played in Einstein’s thinking. Nevertheless, most treatments of special relativity take it as the point of departure and we shall follow this tradition.

12.3 The Michelson–Morley Experiment

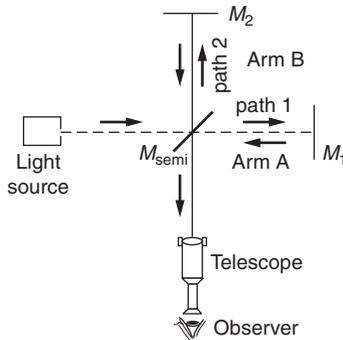
The problem that Michelson attacked was to detect the effect of the Earth’s motion on the speed of light. Briefly, Maxwell’s electromagnetic theory (1861) predicted that electromagnetic disturbances in empty space would propagate at 3×10^8 m/s—the speed of light. The evidence was overwhelming that light consisted of electromagnetic waves, but there was a serious conceptual difficulty.

The only waves then known to physics propagated in matter—solid, liquid, or gas. A sound wave in air, for example, consists of alternate regions of higher and lower pressure propagating with a speed of 330 m/s, somewhat less than the speed of molecular motion. The speed of mechanical waves in a metal bar is higher, typically 5000 m/s. The speed of sound increases with the rigidity of the material or the strength of the “spring forces” between neighboring atoms.

Electromagnetic wave propagation seemed to be fundamentally different. By analogy with mechanical waves in matter, electromagnetic waves were assumed to propagate through space as vibrations in a medium called the *ether* that supported electromagnetic wave propagation. Unfortunately, the ether had to possess contradictory properties; immensely rigid to allow light to propagate at 3×10^8 m/s while so insubstantial that it did not interfere with the motion of the planets.

One consequence of the ether hypothesis is that the speed of light should depend on the observer’s motion relative to the ether. Maxwell suggested an astronomical experiment to detect this effect. The motion of the planet Jupiter through space relative to the Earth should affect the speed with which its light reaches us. The periodic eclipses of the moons of Jupiter create a clock. The clock should appear to periodically advance or fall behind, as the speed of light increases or decreases as Jupiter approaches to and recedes from the Earth. The effect turned out to be too small to be measured accurately. Nevertheless, Maxwell’s proposal was historically important: it stimulated Albert A. Michelson, a young U.S. Navy officer at Annapolis, to invent a laboratory experiment for measuring the Earth’s motion through the ether.

The following explanation of the Michelson–Morley experiment assumes some familiarity with optical interference. If you do not yet know about interference, you can skip the description and take the conclusion on faith: the speed of light is always the same, regardless of the relative motion of the source and the observer.



Michelson’s apparatus was an optical interferometer. As shown in the drawing, light from a source is split into two beams by a semi-silvered mirror M_{semi} that reflects half the light and transmits half. Half of the beam from the light source travels straight ahead on path 1, passing through M_{semi} until it is reflected by mirror M_1 . It then returns to mirror M_{semi} , and half is reflected to the observer. The remainder of the beam from the light source is the half that is reflected by M_2 , along path 2. It is reflected by mirror M_2 , which directs it to the observer after passing again through M_{semi} . Thus beams 1 and 2 each have 1/4 the intensity of the initial beam.

If beams 1 and 2 travel the same distance, they arrive at the observer in phase so that their electric fields add. The observer sees light. However, if the path lengths differ by *half* a wavelength, the fields arrive out of phase and cancel so that no light reaches the observer. In practice, the two beams are slightly misaligned and the observer sees a pattern of bright and dark interference fringes.

If the length of one of the arms is slowly changed, the fringe pattern moves. Changing the difference in path lengths by one wavelength shifts the pattern by one fringe.

The motion of the Earth through the ether should cause a difference between the times for light to transit the two arms of the interferometer, just as if there were a small change in the distance. The difference in transit time depends on the orientation of the arms with respect to the velocity of the Earth through the ether.

We suppose that the laboratory moves through the ether with speed v and that arm A lies in the direction of motion while arm B is perpendicular. According to the ether hypothesis, an observer moving toward the source of a light signal with speed v will observe the signal to travel with speed $c + v$, while for motion away from the source the speed is $c - v$.

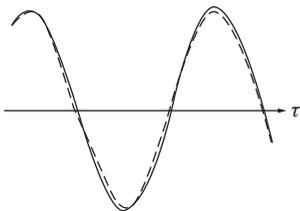
If the length of the arms from the partially silvered mirror M_{semi} to their ends is l , then the time interval for the light to go from M_{semi} to M_1 and return along arm A is τ_A , where

$$\tau_A = \frac{l}{c + v} + \frac{l}{c - v} = \frac{2l}{c} \left(\frac{1}{1 - v^2/c^2} \right).$$

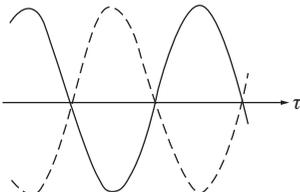
Because $v^2/c^2 \ll 1$, we can simplify the result using the Taylor series expansion in Note 1.3: $1/(1 - x) = 1 + x + x^2 + \dots$. Letting $x = v^2/c^2$ we have

$$\tau_A \approx \frac{2l}{c} \left(1 + \frac{v^2}{c^2} \right).$$

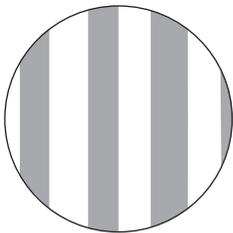
Arm B is perpendicular to the motion so the speed of light is not affected by the motion. However, there is nevertheless a time delay due to the motion of M_{semi} as the light traverses the arm. Denoting the round trip time by τ_B , then during that interval M_{semi} moves a distance $v\tau_B$.

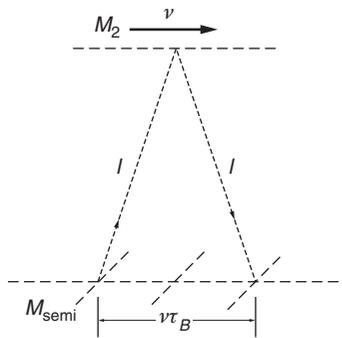


(a) in phase



(b) 180° out of phase





Consequently, the light travels along the hypotenuse of the right triangles shown in the sketch, and the distance traveled is $2\sqrt{l^2 + (v\tau_B/2)^2}$.

Consequently,

$$\tau_B = \frac{2}{c} \sqrt{l^2 + (v\tau_B/2)^2},$$

which gives

$$\tau_B = \frac{2l}{c} \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Using the approximation $1/\sqrt{1-x} = 1 + (1/2)x + (1/8)x^2 + \dots$, and keeping the first term, we have

$$\tau_B = \frac{2l}{c} \left(1 + \frac{1}{2}v^2/c^2 \right).$$

The difference in time for the two paths is

$$\Delta\tau = \tau_A - \tau_B \approx \frac{l}{c} \left(\frac{v^2}{c^2} \right).$$

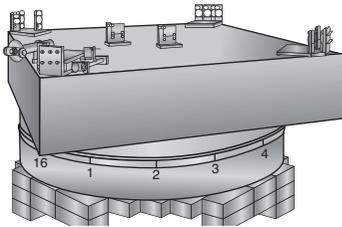
The frequency of light ν is related to its wavelength λ and the speed of light by $\nu = c/\lambda$. The interference pattern shifts by one fringe for each cycle of delay. Consequently, the number of fringe shifts caused by the time difference is

$$N = \nu\Delta\tau = \frac{l}{\lambda} \left(\frac{v^2}{c^2} \right).$$

The orbital speed of the Earth around the Sun gives $v/c \approx 10^{-4}$. Taking the path length $l = 1.2$ m, and using sodium light for which $\lambda = 590 \times 10^{-9}$ m, Michelson predicted a fringe shift of $N = 0.02$. In his initial attempt in 1881, Michelson searched for a fringe shift as the rotation of the Earth changed the direction of motion through the ether, but could detect none to within experimental accuracy.

In 1887 Michelson repeated the experiment in collaboration with the chemist Edward Morley using an apparatus mounted on a granite slab 35 cm thick that floated on mercury and could be continuously rotated. The path length was extended by a factor of 10 using repeated reflections between the mirrors. However, again no fringe shift. The Michelson–Morley experiment has been refined and repeated over the years but no effect of motion through the ether has ever been detected. We are forced to recognize that the speed of light is unaffected by motion of the observer through the ether. Ironically, Michelson, who conceived and executed the experiment for which he is famous, viewed it as a failure. He set out to see the effect of motion through the ether but could not detect any.

Various attempts to explain the null result of the Michelson–Morley experiment introduced such complexity as to threaten the foundations of electromagnetic theory. One attempt was the hypothesis proposed by the



Irish physicist G.F. FitzGerald and the Dutch physicist H.A. Lorentz that motion of the Earth through the ether caused a shortening of one arm of the Michelson interferometer (the “Lorentz–FitzGerald contraction”) by exactly the amount required to eliminate the fringe shift. Other theories that involved such artifacts as drag of the ether by the Earth were even less productive. The elusive nature of the ether remained a troubling enigma.

12.4 The Special Theory of Relativity

It is an indication of Einstein’s genius that the troublesome problem of the ether pointed the way not to complexity and elaboration but to a simplification that unified the fundamental concepts of physics. Einstein regarded the difficulty with the ether not as a fault in electromagnetic theory but as an error in basic dynamical principles. He presented his ideas in the form of two postulates, prefacing them with a note on simultaneity and how to synchronize clocks.

12.4.1 Synchronizing Clocks

Before presenting his theory of space and time, Einstein considered the elementary process of comparing measurements of time by different observers having identical clocks. For the measurements to agree, the clocks must be *synchronized*—they must be adjusted to agree on the time of a single event. In Newtonian physics, if a flash of light occurs, the flash arrives simultaneously at all synchronized clocks, wherever their locations.

The Newtonian procedure would work if the speed of light were infinite or so large that it could be regarded as infinite. However, if one accepts that signals can propagate no faster than the speed of light, the procedure is wrong in principle. For instance, a signal from the Moon to the Earth takes about one second. One might attempt to synchronize a clock on the Moon with a clock on the Earth by advancing the Moon clock by one second. With this adjustment, the Moon clock would always appear to agree with the Earth clock. However, for the observer on the Moon, the Earth clock would always lag the Moon clock by two seconds. Thus the clocks would be synchronized for one observer but not the other.

Einstein proposed a simple procedure for synchronizing clocks so that all observers agree on the time of an event. Observer A sends observer B a signal at time T_A . Observer B notes that the signal arrives at time T_B on the local clock. B immediately sends a signal back to A who detects it at time $T'_A = T_A + \Delta T$. The clocks are synchronized if B ’s clock reads $T_B = T_A + \Delta T/2$. Interpreting the times reported by different observers requires knowing their positions, but everyone would agree on the time of an event.

Einstein thought about time measurements in terms of railway clocks at different stations for which light propagation times are of no

practical importance. Today, Einstein's procedure for synchronizing clocks is crucially important: it is essential for comparing atomic clocks in international time standards laboratories, as well as for keeping the Internet synchronized and for maintaining the voltage–current phase across the national power grid.

12.4.2 The Principle of Relativity

The special theory of relativity rests on two postulates. The first, known as the principle of relativity, is that the laws of physics have the same form with respect to all inertial systems. In Einstein's words: "The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion." The principle of relativity was hardly novel; Galileo is credited with first pointing out that there is no dynamical way to determine whether one is moving uniformly or is at rest, and Newton gave it rigorous expression in his dynamical laws in which acceleration, not velocity, is paramount. If the principle of relativity were not true, energy and momentum might be conserved in one inertial system but not in another. The principle of relativity played only a minor role in the development of classical mechanics: Einstein elevated it to a keystone of dynamics. He extended the principle to include not only the laws of mechanics but also the laws of electromagnetic interaction and *all* the laws of physics. Furthermore, in his hands the principle of relativity became a powerful tool for discovering the correct form of physical laws.

We can only guess at the sources of Einstein's inspiration, but they must have included the following consideration. If the speed of light were not a universal constant, that is, if the ether could be detected, then the principle of relativity would fail; a special inertial frame would be singled out, the one at rest in the ether. However, the form of Maxwell's equations, as well as the failure of any experiment to detect motion through the ether, cause us to conclude that the speed of light is independent of the motion of the source. Our inability to detect absolute motion, either with light or with Newtonian dynamics, forces us to accept that absolute motion has no role in physics.

The Universal Speed

The second postulate of relativity is that the speed of light is a universal constant, the same for all observers. "Any ray of light moves in the stationary system of co-ordinates with the determined speed c , whether the ray be emitted by a stationary or by a moving body." Einstein argued that because the speed of light c predicted by electromagnetic theory involves no reference to a medium, then no matter how we measure the speed of light the result will *always* be c , independent of our motion. This is in contrast to the behavior of sound waves, for example, where

the observed speed of the wave depends on the motion of the observer through the medium. The idea of a universal speed was indeed a bold hypothesis, contrary to all previous experience and, for many of Einstein's contemporaries, defying common sense. But common sense can be a poor guide. Einstein once quipped that common sense consists of the prejudices one learns before the age of eighteen.

Rather than regarding the absence of the ether as a paradox, Einstein saw that the concept of a universal speed preserved the simplicity of the principle of relativity. His view was essentially conservative; he insisted on maintaining the principle of relativity that the ether would destroy. The urge toward simplicity appeared to be fundamental to Einstein's personality. The special theory of relativity was the simplest way to preserve the unity of classical physics.

To summarize, the postulates of special relativity are: *The laws of physics have the same form in all inertial systems. The speed of light in empty space is a universal constant, the same for all observers regardless of their motion.*

These postulates require us to revise our ideas about space and time, and this has immediate consequences for physics. The mathematical expression of kinematics and dynamics in the special theory of relativity is embodied in the Lorentz transformation—a simple prescription for relating events in different inertial systems.

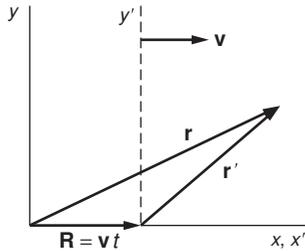
12.5 Transformations

In the world of relativity, a transformation is a set of equations that relate observations in one coordinate system to observations in another. As you will see, the logic of special relativity is reasonably straightforward and the mathematics is not arcane. Nevertheless, the reasoning is likely to seem perplexing because of the underlying question “Isn't this a peculiar way to do physics?” The answer is “Yes! This is a most peculiar way to do physics!” Rather than examining forces, conservation laws, dynamical equations, and other staples of Newtonian physics, Einstein merely discussed how things look to different observers.

Einstein was the first person to use transformation theory to discover new physical behavior, in particular, to create the theory of special relativity. From two simple assumptions, he derived a new way to look at space and time and discovered a new system of dynamics.

Special relativity can be written with all the elegance of a beautiful mathematical theory but its most attractive attribute is that it not only looks beautiful, it works beautifully. The theory of special relativity is among the most carefully studied theories in physics and its predictions have always been correct within experimental error.

The heart of special relativity is the Lorentz transformation, but to introduce Einstein's approach let us first look at the corresponding procedure for Newtonian physics where the transformation is known as the Galilean transformation.



12.5.1 The Galilean Transformation

We will frequently refer to observations in two standard inertial systems: $S = (x, y, z, t)$ and $S' = (x', y', z', t')$. S' moves with respect to S at speed v in the x direction. Alternatively, S moves with respect to S' at speed v in the negative x direction. For convenience, we take the origins to coincide at $t = 0$, and take the x and x' axes to be parallel.

If a particular point in space has coordinates $\mathbf{r} = (x, y, z)$ in S , the coordinates in S' are $\mathbf{r}' = (x', y', z')$. These are related by

$$\mathbf{r}' = \mathbf{r} - \mathbf{R},$$

where

$$\mathbf{R} = \mathbf{v}t.$$

Since \mathbf{v} is in the x direction, we have

$$\begin{aligned} x' &= x - vt, \\ y' &= y \\ z' &= z \\ t' &= t. \end{aligned} \tag{12.1}$$

The fourth equation $t' = t$, listed merely for completeness, is taken for granted in Newtonian dynamics, and follows immediately from the Newtonian concept of “ideal” time.

Equations (12.1) are known as the *Galilean transformation*. Because the laws of Newtonian mechanics hold in all inertial systems, the form of the laws is unaffected by this transformation. More concretely, there is no way to distinguish between systems on the basis of the motion they predict. The following example illustrates what this means.

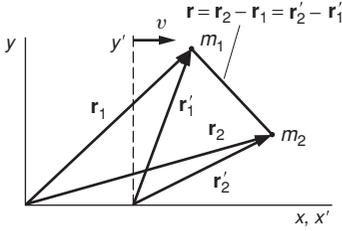
Example 12.1 Applying the Galilean Transformation

Consider how we might discover the law of force between two isolated bodies from observations of their motion. For example, the problem might be to discover the law of gravitation from data on the elliptical orbit of one of Jupiter’s moons. If m_1 and m_2 are the masses of the moon and of Jupiter, respectively, and if \mathbf{r}_1 and \mathbf{r}_2 are their positions relative to an observer on the Earth, we have

$$\begin{aligned} m_1 \ddot{\mathbf{r}}_1 &= \mathbf{F}(r) \\ m_2 \ddot{\mathbf{r}}_2 &= -\mathbf{F}(r), \end{aligned}$$

where we assume that the force \mathbf{F} between the bodies is a central force that depends only on the separation $r = |\mathbf{r}_2 - \mathbf{r}_1|$. From our observations of $\mathbf{r}_1(t)$ we can evaluate $\ddot{\mathbf{r}}_1$, from which we obtain the value of \mathbf{F} . Suppose the data reveal that $\mathbf{F}(r) = -Gm_1m_2\hat{\mathbf{r}}/r^2$.

Now let us look at the problem from the point of view of an observer in a spacecraft that is moving with constant speed far from the Earth.



According to the principle of relativity this observer must obtain the same force law as the earthbound observer. The situation is represented in the drawing. x, y is the earthbound system, x', y' is the spacecraft system, and \mathbf{v} is the relative velocity of the two systems. Note that the vector \mathbf{r} from m_1 to m_2 is the same in both coordinate systems.

In the x', y' system the observer sees that the moon is accelerating at rate $\ddot{\mathbf{r}}'_1$ and concludes that the force is

$$\mathbf{F}'(r) = m_1 \ddot{\mathbf{r}}'_1.$$

A fundamental property of the Galilean transformation is that acceleration is unaltered. Here is the formal proof: because $\dot{\mathbf{v}} = 0$, we have

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}'_1 + \mathbf{v}t \\ \dot{\mathbf{r}}_1 &= \dot{\mathbf{r}}'_1 + \mathbf{v} \\ \ddot{\mathbf{r}}_1 &= \ddot{\mathbf{r}}'_1. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbf{F}'(r') &= m_1 \ddot{\mathbf{r}}'_1 \\ &= m_1 \ddot{\mathbf{r}}_1 \\ &= \mathbf{F}(r) \\ &= -\frac{Gm_1m_2}{r^2} \hat{\mathbf{r}}. \end{aligned}$$

The law of force is identical in the two systems. This is what we mean when we say that two inertial systems are equivalent. If the form of the law, or the value of G , were not identical we could make a judgment about the speed of a coordinate system in empty space by investigating the law of gravitation in that system. The inertial systems would not be equivalent.

Example 12.1 is almost trivial because the force depends on the separation of the two particles, a quantity that is unchanged (invariant) under the Galilean transformation. All forces in Newtonian physics are due to interactions between particles, interactions that depend on the *relative* coordinates of the particles. Consequently, they are invariant under the Galilean transformation.

What happens to the equation for a light signal under the Galilean transformation? The following example shows the difficulty.

Example 12.2 Describing a Light Pulse by the Galilean Transformation

At $t = 0$ a pulse of light is emitted from the origin of the S system, and travels along the x axis at speed c . The equation for the location of the pulse along the x axis is $x = ct$.

In the S' system, the equation for the wavefront along the x' axis is

$$\begin{aligned}x' &= x - vt \\ &= (c - v)t,\end{aligned}$$

where v is the relative velocity of the two systems. The speed of the pulse in the S' system is

$$\frac{dx'}{dt} = c - v.$$

But this result is contrary to the postulate that the speed of light is always c , the same for all observers.

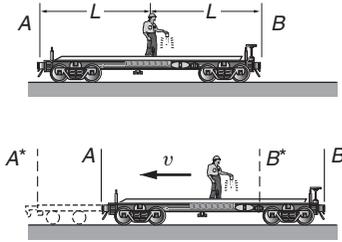
Because the Galilean transformation is incompatible with the principle that the speed of light is always c , our task is to find a transformation that is compatible. Before undertaking this, it is useful to think carefully about the nature of measurement.

The Galilean transformation relates the spatial coordinates of an event measured by observers in two inertial systems moving with relative speed v . By an “event” we mean the unique values of a set of coordinates in space and time. Physically meaningful measurements invariably involve more than a single event. For instance, measuring the length of a rod involves placing the rod along a calibrated scale such as a meter stick, and recording the position at each end. Consequently, length involves two measurements. If the rod is at rest along the axis in the S system, the coordinates of its end points might be x_a and x_b , where $x_b = x_a + L$. According to an observer in the S' system, the x' coordinates are given by Eq. (12.1): $x'_a = x_a - vt$ and $x'_b = x_b - vt$. Since $x_b = x_a + L$, we have $L = x_b - x_a$ and $L' = x'_b - x'_a = L$. The two observers agree on the length.

In this simple exercise in measurement we have made a natural assumption: the measurements are made *simultaneously*. This is not important in the S system because the rod is at rest. However, in the S' system the rod is moving. If the end points were recorded at different times, the value for L' would have been incorrect. We have used the Galilean assumption $t' = t$, which implies that if measurements are simultaneous in one coordinate system they are simultaneous in all coordinate systems. This would be the case if the speed of light were infinite, but the finite speed of light profoundly affects our idea of simultaneity. We therefore digress briefly to examine the nature of simultaneity.

12.6 Simultaneity and the Order of Events

We have an intuitive idea of simultaneity: two events are simultaneous if their time coordinates have the same value. However, as the following example shows, events that are simultaneous in one coordinate system are not necessarily simultaneous when observed in a different coordinate system.



Example 12.3 Simultaneity

A railwayman stands at the middle of a flatcar of length $2L$. He flicks on his lantern and a light pulse travels out in all directions with the velocity c .

Light arrives at the ends of the car after a time interval L/c . In this system, the flatcar's rest system, the light arrives simultaneously at the end points A and B .

Now let us observe the same situation from a frame moving to the right with velocity v . In this frame the flatcar moves to the left with velocity v . As observed in this frame the light still has velocity c , according to the second postulate of special relativity. However, during the transit time, A moves to A^* and B moves to B^* . It is apparent that the pulse arrives at B^* before A^* ; the events are not simultaneous in this frame.

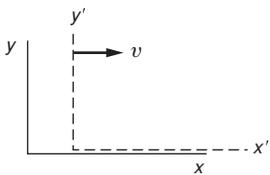
Just as events that are simultaneous in one inertial system may not be simultaneous in another, it can be shown that events that are spatially coincident—having the same coordinates in space—in one system may not appear to be coincident in another. We shall show later that two events can be classified as either *spacelike* or *timelike*. For spacelike events it is impossible to find a coordinate system in which the events coincide in space, though there is a system in which they are simultaneous in time. For timelike events it is impossible to find a coordinate system in which the events are simultaneous in time, though there is a system in which they coincide in space.

At this point we need a systematic way to solve the problem of relating observations made in different inertial systems in a fashion that obeys the principle of relativity. This task constitutes the core of special relativity.

12.7 The Lorentz Transformation

The failure of the Galilean transformation to satisfy the postulate that the speed of light is a universal constant constituted a profound dilemma. Einstein solved the dilemma by introducing a new transformation law for relating the coordinates of events as observed in different inertial systems. He introduced a system designed to ensure that a signal moving at the speed of light in one system would be observed to move at the same speed in the other, irrespective of the relative motion. Such a “fix” took some courage because to alter a transformation law is to alter the fundamental relation between space and time.

Let us refer once more to our standard systems, the S system (x, y, z, t) , and the S' system (x', y', z', t') . The system S' moves with velocity v along the positive x axis, and the origins coincide at $t = t' = 0$. We take the most general transformation relating the coordinates of a given event



in the two systems to be of the form

$$x' = Ax + Bt \quad (12.2a)$$

$$y' = y \quad (12.2b)$$

$$z' = z \quad (12.2c)$$

$$t' = Cx + Dt. \quad (12.2d)$$

Some comments on Eqs. (12.2): the transformation equations are linear because a nonlinear transformation could yield an acceleration in one system even if the velocity were constant in the other. Further, we leave the y' and z' axes unchanged, by symmetry.

Here is one model to justify the assumption that $y' = y$ and $z' = z$: consider two trains on parallel tracks. Each train has an observer holding a paint brush at the same height in their system, say at 1 m above the floor of the train. Each train is close to a wall. The trains approach at relative speed v , and each observer holds the brush to the wall, leaving a stripe. Observer 1 paints a blue stripe and observer 2 paints a yellow stripe.

Suppose that observer 1 sees that the height of observer 2 has changed, so that the blue stripe is below the yellow stripe. Observer 2 would have to see the same phenomenon except that it is now the yellow stripe that is below the blue stripe. Because their conclusions are contradictory they cannot both be right. Since there is no way to distinguish between the systems, the only conclusion is that both stripes are at the same height. We conclude that distance perpendicular to the direction of motion is unchanged by the motion of the observer.

We can evaluate the four constants A, B, C, D in Eqs. (12.2) by comparing coordinates for four events. These could be:

(1) The origin of S is observed in S' :

$$S : (x = 0, t); \quad S' : (x' = -vt', t').$$

From Eqs. (12.2a) and (12.2d), $-vt' = 0 + Bt$, and $t' = 0 + Dt$.

Consequently, $B = -vD$.

(2) The origin of S' is observed in S :

$$S' : (x' = 0, t'); \quad S : (x = +vt, t).$$

From Eq. (12.2a), $0 = Avt + Bt$.

Consequently, $B = -vA$, and using result (1), it follows that

$$D = A.$$

(3) A light pulse is emitted from the origin at $t = 0, t' = 0$ and is observed later along the x and x' axes.

$$S : (x = ct, t); \quad S' : (x' = ct', t').$$

From Eqs. (12.2a) and (12.2d), $ct' = ctA + Bt$, and $t' = ctC + Dt$

Using $D = A$ and $B = -vA$, it follows that

$$C = -(v/c^2)A.$$

(4) A light pulse is emitted from the origin at $t = 0, t' = 0$ and is observed later along the y axis in the S system. In S , $x = 0, y = ct$, but in S' the pulse has both x' and y' coordinates.

$$S : (x = 0, y = ct, t); \quad S' : (x' = -vt', y' = \sqrt{(ct)^2 - (-vt')^2}, t').$$

From Eqs. (12.2b) and (12.2d), $ct = \sqrt{(ct')^2 - (-vt')^2}$ and $t' = Dt$ which give

$$D = 1/\sqrt{1 - v^2/c^2}.$$

(We selected the positive sign for the square root because otherwise t and t' would have opposite signs.) The factor $1/\sqrt{1 - v^2/c^2}$ occurs so frequently that it is given a special symbol:

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Note that $\gamma \geq 1$ and that as $v \rightarrow c$, $\gamma \rightarrow \infty$.

Substituting our results in Eqs. 12.2 yields

$$x' = \gamma(x - vt) \tag{12.3a}$$

$$y' = y \tag{12.3b}$$

$$z' = z \tag{12.3c}$$

$$t' = \gamma(t - vx/c^2). \tag{12.3d}$$

The transformation from S' to S can be found by letting $v \rightarrow -v$:

$$x = \gamma(x' + vt') \tag{12.4a}$$

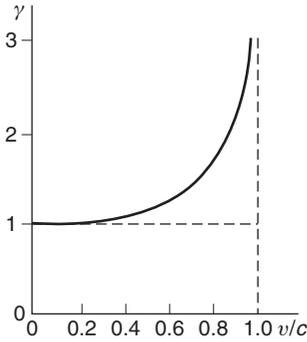
$$y = y' \tag{12.4b}$$

$$z = z' \tag{12.4c}$$

$$t = \gamma(t' + vx'/c^2). \tag{12.4d}$$

Equations (12.3) and (12.4) are the prescription for relating the coordinates of an event in different inertial systems so as to satisfy the postulates of special relativity. They are called the *Lorentz transformation* after the physicist Hendrik Lorentz who first wrote them, though in a very different context.

The Lorentz transformation equations have a straightforward physical interpretation. The factor γ is a scaling factor that ensures that the speed of light is the same in both systems. The factor vt in Eq. (12.3a) reveals that system S' is moving in the positive x direction, with speed v . The factor vx/c^2 in Eq. (12.3d) is a little more subtle. The clock synchronization algorithm requires that the time registered on a clock be corrected for the transit time τ_{transit} from the event point. If the point is moving with speed v , then the transit time correction must be adjusted correspondingly. The additional distance traveled is $d = v\tau_{\text{transit}}$, where $\tau_{\text{transit}} = x/c$. Hence, the time in Eq. (12.3d) needs to be corrected by the quantity $d/c = vx/c^2$.



In the limit $v/c \rightarrow 0$ (or alternatively $c \rightarrow \infty$), where $\gamma \rightarrow 1$, the Lorentz transformation becomes identical to the Galilean transformation. However, in the general case, the Lorentz transformation requires a rethinking of the concepts of space and time.

Before looking into the consequences of this rethinking, let us examine how the Lorentz transformation demonstrates why Michelson's experiment had to give a null result.

12.7.1 Michelson–Morley Revisited

With the Lorentz transformation in hand we can understand why the Michelson–Morley experiment failed to display any fringe shift as the apparatus was rotated. We introduce again the two reference systems $S : (x, y, z, t)$ and the system $S' : (x', y', z', t')$ moving with relative speed v along x . Their origins coincide at $t = t' = 0$. A pulse of light is emitted at $t = 0$ in the S system and spreads spherically. The locus of the pulse is given by $x^2 + y^2 + z^2 = (ct)^2$. We leave it as an exercise to show that the Lorentz transformation, Eqs. (12.3), predicts that in the “moving” S' system the locus of the pulse is given by $x'^2 + y'^2 + z'^2 = (ct')^2$. Observers in the two frames see the same phenomenon: a pulse spreading in space with the speed of light c . There is no trace of a reference to their relative speed v .

The Michelson–Morley experiment was designed to show the difference in the speed of light between directions parallel and perpendicular to the Earth's motion but according to the second postulate of the special theory of relativity—the speed of light is the same for all observers—there should be none. The Lorentz transformation shows explicitly that there is none.

12.8 Relativistic Kinematics

Because the principles of special relativity require us to rethink basic ideas of measurement and observation, they have important consequences for dynamics. The goal for the rest of this chapter is to learn how the principles of special relativity are employed to relate measurements in different inertial systems. The motivation for this is to some extent practical: relativistic kinematics is essential in areas of physics ranging from elementary particle physics to cosmology and also to technologies such as the global positioning system. More fundamentally, the study of relativistic transformations leads to new physics, most famously the relation $E = mc^2$, and to an elegant unified approach to dynamics and electromagnetic theory.

Predictions of the Lorentz transformation often defy intuition because we lack experience moving at speeds comparable to the speed of light. Two surprising predictions are that a moving clock runs slow and a moving meter stick contracts. These follow from the Lorentz transformation of time and space intervals. We will also derive these results by

geometric arguments that may help provide intuition about this unexpected behavior.

Caution: in the discussion to follow, *either* S or S' may be the rest system for an observer, and in addition there is the possibility of introducing other systems. We need to be clear not only about the physical phenomena taking place but the system from which it is being observed.

12.8.1 Time Dilation

A clock is at rest in S at some location x . The clock's rate is determined by the interval τ_0 between its ticks. The problem is to find the corresponding interval observed in the S' system, in which the clock is moving with speed $-v$.

Successive ticks of the clock in the rest system S are

$$\text{tick 1 (event 1) : } t$$

$$\text{tick 2 (event 2) : } t + \tau_0.$$

The corresponding times observed in the moving system S' are, from Eq. (12.3d),

$$t' = \gamma(t - vx/c^2)$$

$$t' + \tau'_0 = \gamma(t + \tau_0 - vx/c^2).$$

Subtracting, we obtain

$$\tau'_0 = \gamma\tau_0. \quad (12.5)$$

Because $\gamma \geq 1$, the time interval observed in the moving system is longer than in the clock's rest system. Thus, the moving clock runs slow. As $v \rightarrow c$, time stands still.

This result, known as *time dilation*, is hardly intuitive and so it may be instructive to derive it by a different approach.

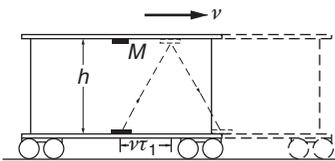
Let us consider an idealized clock in which the timing element consists of two parallel mirrors with a light pulse bouncing between them. (Our discussion follows *Introduction to Electrodynamics*, David J. Griffiths, Prentice Hall, Upper Saddle Ridge New Jersey, 1999.)

Each round trip of the light constitutes a clock tick. The clock is mounted vertically on a railway car that moves with speed v , as shown. An observer on the railway car monitors the rate of the ticks. If the distance between the mirrors is h , then the time interval between ticks is

$$\tau_0 = 2h/c.$$

In this calculation, the railway car is the rest system S for the clock.

An observer on the ground system S' also monitors the rate of ticks of the clock. S' is moving at speed $-v$ with respect to the rest system on the railway car. For this observer, the time interval for the light, up or down,



is $\tau_1 = \sqrt{h^2 + (v\tau_1)^2}/c$. Solving for τ_1 , the roundtrip time τ'_0 is

$$\tau'_0 = 2\tau_1 = (2h/c) \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Recalling that $\gamma = 1/\sqrt{1 - v^2/c^2}$, we have

$$\tau'_0 = \gamma\tau_0$$

in agreement with Eq. (12.5).

Example 12.4 The Role of Time Dilation in an Atomic Clock

Possibly you have looked through a spectroscope at the light from an atomic discharge lamp. Each line of the spectrum is the light emitted when an atom makes a transition between two of its internal energy states. The lines have different colors because the frequency ν of the light is proportional to the energy change ΔE in the transition. If ΔE is of the order of electron-volts, the emitted light is in the optical region ($\nu \approx 10^{15}$ Hz). There are some transitions, however, for which the energy change is so small that the emitted radiation is in the microwave region ($\nu \approx 10^{10}$ Hz). These microwave signals can be detected and amplified with available electronic instruments. Since the oscillation frequency depends almost entirely on the internal structure of the atom, the signals can serve as a frequency reference to govern the rate of an atomic clock. Atomic clocks are highly stable and relatively immune to external influences.

Each atom radiating at its natural frequency serves as a miniature clock. The atoms are frequently in a gas and move randomly with thermal velocities. Because of their thermal motion, the clocks are not at rest with respect to the laboratory and the observed frequency is shifted by time dilation.

Consider an atom that is radiating its characteristic frequency ν_0 in the rest frame. We can think of the atom's internal harmonic motion as akin to the swinging motion of the pendulum of a grandfather clock: each cycle corresponds to a complete swing of the pendulum. If the period of the swing is τ_0 seconds in the rest frame, the period in the laboratory is $\tau = \gamma\tau_0$. The observed frequency in the laboratory system is

$$\begin{aligned} \nu &= \frac{1}{\tau} = \frac{1}{\gamma\tau_0} = \frac{\nu_0}{\gamma} \\ &= \nu_0 \sqrt{1 - \frac{v^2}{c^2}}. \end{aligned}$$

The shift in the frequency is $\Delta\nu = \nu - \nu_0$. If $v^2/c^2 \ll 1$, $\gamma \approx 1 - \frac{1}{2}v^2/c^2$, and the fractional change in frequency is

$$\frac{\Delta\nu}{\nu_0} = \frac{\nu - \nu_0}{\nu_0} = -\frac{1}{2} \frac{v^2}{c^2}. \quad (1)$$

A handy way to evaluate the term on the right is to multiply numerator and denominator by the mass of the atom M :

$$\frac{\Delta\nu}{\nu_0} = -\frac{\frac{1}{2}M\bar{v}^2}{Mc^2}.$$

$\frac{1}{2}M\bar{v}^2$ is the kinetic energy due to thermal motion of the atom. This energy increases with the temperature of the gas, and according to our treatment of the ideal gas in Section 5.9

$$\frac{1}{2}M\bar{v}^2 = \frac{3}{2}kT,$$

where \bar{v}^2 is the average squared velocity, $k = 1.38 \times 10^{-23}$ J/deg is Boltzmann's constant, and T is the absolute temperature.

In the atomic clock known as the hydrogen maser, the reference frequency arises from a transition in atomic hydrogen. M is close to the mass of a proton, 1.67×10^{-27} kg, and using $c = 3 \times 10^8$ m/s, we find

$$\begin{aligned} \frac{\Delta\nu}{\nu} &= \frac{\frac{3}{2}kT}{Mc^2} = -\frac{\frac{3}{2}(1.38 \times 10^{-23})T}{(1.67 \times 10^{-27})(9 \times 10^{16})} \\ &= 1.4 \times 10^{-13}T. \end{aligned}$$

At room temperature, $T = 300$ K (300 degrees on the absolute temperature scale $\approx 27^\circ\text{C}$), we have

$$\frac{\Delta\nu}{\nu} = -4.2 \times 10^{-11}.$$

This is a sizable effect in modern atomic clocks. In order to correct for time dilation to an accuracy of 1 part in 10^{13} , it is necessary to know the temperature of the hydrogen atoms to an accuracy of 1 K. However, if one wishes to compare frequencies to parts in 10^{15} , the absolute temperature must be known to within a millikelvin, a much harder task.

The creation of techniques to cool atoms to the microkelvin regime has opened the way to a new generation of atomic clocks. These clocks, operating at optical rather than microwave frequencies, have achieved a stability greater than 1 part in 10^{17} —equivalent to a difference of about 1 second over the age of the Earth.

12.8.2 Length Contraction

A rod at rest in S has length L_0 . What is the length observed in the system S' that is moving with speed $-v$ along the direction of the rod?

The rod lies along the x axis and its ends are at x_a and x_b , where $x_b = x_a + L_0$. The measurement involves two events but because the rod is at rest in S the times are unimportant, so we can take the observations in S to be simultaneous at time t . The length in S is found from the coordinates

of two events:

event 1 : (x_a, t)

event 2 : (x_b, t) .

The length observed in the rest frame S is $x_b - x_a = L_0$. The problem is to find the length observed in system S' where the rod is moving with speed $-v$.

A natural, but *wrong!*, approach to finding the coordinates in S' would be to use Eq. (12.3a) to find values for x'_b and x'_a and subtract. This would give $L'_0 = x'_b - x'_a = \gamma L_0$. The result is wrong because the times for the two events in S' are not identical, as can be seen from Eq. (12.3d).

Meaningful measurements of the dimensions of a moving object must be made simultaneously. We must therefore find the correspondence between values of x' and x at the same time t' in the S' system. This is readily accomplished by applying the Lorentz transformation to relate events in S to those in S' . Equation (12.4a) gives $x = \gamma(x' - vt')$. Consequently,

$$x_b = \gamma(x'_b - vt')$$

$$x_a = \gamma(x'_a - vt')$$

Subtracting, we obtain $x_b - x_a = \gamma(x'_b - x'_a)$, so that $L_0 = \gamma L'$ and

$$L' = L_0/\gamma = \sqrt{1 - v^2/c^2}L_0. \quad (12.6)$$

The rod appears to be contracted. As $v \rightarrow c$, $L' \rightarrow 0$. The contraction occurs along the direction of motion only: if the rod lay along the y axis, we would use the transformation $y' = y$ and conclude that $L' = L_0$.

As in the case of time dilation, we have a non-intuitive result. This, too, can be understood using a geometrical argument.

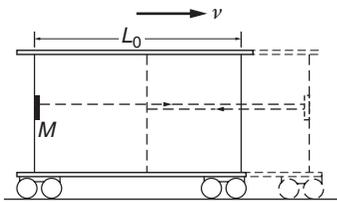
An observer on a train could measure the length of the train car L_0 by bouncing light between mirrors at each end and measuring the roundtrip time τ_0 :

$$\tau_0 = 2\frac{L_0}{c}.$$

The observer on board concludes that the length of the car is

$$L_0 = \frac{c}{2}\tau_0. \quad (12.7)$$

An observer on the ground also measures the length of the car L' as the train goes by at speed $+v$ by measuring the time for a pulse to make a roundtrip between the ends. As seen by the ground observer, the time τ_+ for the pulse to travel from the rear mirror to the front is longer than L'/c , because the front mirror moves slightly ahead during the transit time. The distance traveled is $L' + v\tau_+$. Consequently, $\tau_+ = (L' + v\tau_+)/c$ so that $\tau_+ = L'/(c - v)$. Similarly, the time for the return trip is $\tau_- = L'/(c + v)$.



The roundtrip for the light pulse is

$$\tau'_0 = \tau_+ + \tau_- = L' \left(\frac{1}{c-v} + \frac{1}{c+v} \right) = \frac{2L'}{c} \left(\frac{1}{1-v^2/c^2} \right).$$

Consequently,

$$L' = \frac{c}{2} \tau'_0 (1 - v^2/c^2).$$

Comparing this with Eq. (12.7), we have

$$L' = L_0 \frac{\tau_0}{\tau'_0} (1 - v^2/c^2).$$

Taking the value of τ_0/τ'_0 from Eq. (12.5), we have

$$L' = L_0 \sqrt{1 - v^2/c^2}.$$

Because $L' < L_0$, the ground observer sees the length of the car contracted by the factor $\sqrt{1 - v^2/c^2}$.

12.8.3 Proper Time and Proper Length

We introduced the symbols τ_0 and L_0 to denote time and length intervals observed in the rest frame of the events. These quantities are referred to as *proper*: τ_0 is the *proper time* and L_0 is the *proper length*.

Proper time τ is the time measured by a clock in its own rest system, which might for example be a clock carried aboard a spacecraft. According to Eq. (12.5), a time interval $\Delta t'$ measured in a moving frame is always *greater* than the proper time interval $\Delta \tau$:

$$\Delta t' = \gamma \Delta \tau = \frac{\Delta \tau}{\sqrt{1 - v^2/c^2}} \geq \Delta \tau.$$

Similarly, proper length is the length of an object measured in its own rest frame, for example a meter stick carried aboard a spacecraft. According to Eq. (12.6), the length L' measured in a moving frame is always *less* than the proper length L_0 :

$$L' = \frac{L_0}{\gamma} = \sqrt{1 - v^2/c^2} L_0 \leq L_0.$$

12.8.4 Are Relativistic Effects Real?

Time and distance are such intuitive concepts that it may be difficult, at least at first, to accept that the predictions of special relativity are “real” in the familiar sense of physical reality. We shall look at some examples where time dilation and length contraction unquestionably occur. Paradoxes immediately come to mind, for instance the pole-vaulter paradox: a farmer has a barn with a door at each end. A pole-vaulter runs through the barn gripping a horizontal pole longer than the barn. The farmer wants to slam the doors with the pole inside. The farmer instructs the runner to go so fast that the length contraction permits the pole to fit.

The moment the runner is inside, the farmer slams the doors. The paradox is that from the runner's point of view, the pole is unchanged but the length of the barn has contracted. Rather than making the task of fitting in the barn easier, running makes it harder!

The paradox hinges on the difference between Newtonian and relativistic concepts of simultaneity. The runner will not agree that the doors were both shut at the same time, and it will be left as a problem to show that from the runner's point of view the pole was never totally in the barn.

The first dramatic experimental demonstration of time dilation occurred in an early study of cosmic rays. The experiment also demonstrated that although time dilation and length contraction appear to be fundamentally different phenomena, they are essentially two sides of the same coin.

Example 12.5 Time Dilation, Length Contraction, and Muon Decay

The negatively charged muon (symbol μ^-) is an elementary particle related to the electron: it carries one unit of negative charge, same as the electron, and it has a positively charged antiparticle μ^+ analogous to the positron, the electron's antiparticle. The muon differs from the electron most conspicuously in its mass, which is about 205 times the electron's mass, and in being unstable. Electrons are totally stable, but the muon decays into an electron and two neutrinos.

The decay of the muon is typical of radioactive decay processes: if there are $N(0)$ muons at $t = 0$, the number at time t is

$$N(t) = N(0)e^{-t/\tau}$$

where τ is a time constant characteristic of the decay. It is easy to show that the average time before a given muon decays is τ , and so τ is known as the "lifetime" of the particle. For muons, $\tau = 2.2 \mu\text{s}$. (Caution: the symbol μ stands for "micro", 10^{-6} , as well as for muon. One needs to keep one's wits about symbols in physics.) If the muons travel with speed v , the average distance they travel before decaying is $\langle L \rangle = v\tau$.

Muons were discovered in research on cosmic rays. They are created at high altitudes by high energy protons streaming toward the Earth. The protons are quickly lost in the atmosphere by collisions, but the muons continue to sea level with very little loss. The early experiments ran into a paradox. If one assumes that the muons travel at high speed, close to the speed of light, then the maximum average distance that they travel before decay should be no bigger than $\langle L \rangle = c\tau$. Consequently, after traveling distance L , the flux of muons should be decreased by a factor of at least $\exp -L/\langle L \rangle$. For a $2.2 \mu\text{s}$ lifetime, $\langle L \rangle = 660 \text{ m}$. In the initial experiment (B. Rossi and D. B. Hall, *Physical Review*, 59, 223 (1941)), the flux was monitored on a mountain top in Colorado and

at a site 2000 m below. The flux was expected to decrease by a factor of $\exp -2000 \text{ m}/660 \text{ m} = 0.048$. However, the observed loss ratio was much smaller. The paradox was that the muons behaved as if on their journey to Earth they lived for about three times their known lifetime.

The resolution of the paradox was the realization that the muons actually lived that long. The quantity γ was determined from measurements of the muon energy. When time dilation was taken into account, the lifetimes were calculated to increase by a factor close to the observed value.

The concept of proper time is another way to look at this. The muons carry their own “clock” that determines their decay rate. Their clock, in the muon rest frame, measures proper time, so the decay rate measured by a ground observer is longer. Using modern particle accelerators, muons can be created with much higher energies than obtained with cosmic rays, leading to correspondingly larger values of γ . In one experiment (R. M. Casey, *et al.*, Phys. Rev. Letters, 82, 1632 (1997)) the lifetime was extended so much that useful signals could be observed for up to 440 μs , 200 times the muon lifetime. Do moving clocks “really” run slow? The answer depends on how you wish to interpret the experiment. In a coordinate system moving with the muons (in the muon rest system), the particles decay with their natural decay rate. However, in this system the muons “see” that the thickness of the atmosphere is smaller than seen by a ground-based observer. Lorentz contraction reduced the path length from 2000 m to $2000/\gamma$ m. The fraction of muons that penetrated through is the same as if the problem were viewed from a ground-based coordinate system.

We see that once we accept the postulates of relativity we are forced to abandon the intuitive idea of simultaneity. Nevertheless, the Lorentz transformation, which embodies the postulates of relativity, allows us to calculate the times of events in two different systems.

Example 12.6 An Application of the Lorentz Transformation

A light pulse is emitted at the center of a railway car $x = 0$ at time $t = 0$. How do we find the time of arrival of the light pulse at each end of the railway car, which has length $2L$? The problem is trivial in the rest frame. The two events are

$$\begin{aligned} \text{Event 1: The pulse arrives at end A} & \left\{ \begin{array}{l} x_1 = -L \\ t_1 = \frac{L}{c} = T \end{array} \right. \\ \text{Event 2: The pulse arrives at end B} & \left\{ \begin{array}{l} x_2 = L \\ t_2 = \frac{L}{c} = T. \end{array} \right. \end{aligned}$$

To find the time of the events in system S' moving with respect to the railway car, we apply the Lorentz transformation for the time coordinates.

Event 1:

$$\begin{aligned} t'_1 &= \gamma \left(t_1 - \frac{vx_1}{c^2} \right) \\ &= \gamma \left(T + \frac{vL}{c^2} \right) \\ &= \frac{1}{\sqrt{1 - v^2/c^2}} \left(T + \frac{v}{c} T \right) \\ &= T \sqrt{\frac{1 + v/c}{1 - v/c}}. \end{aligned}$$

Event 2:

$$\begin{aligned} t'_2 &= \gamma \left(t_2 - \frac{vx_2}{c^2} \right) \\ &= T \sqrt{\frac{1 - v/c}{1 + v/c}}. \end{aligned}$$

In the moving system, the pulse arrives at B (event 2) earlier than it arrives at A , as we anticipated.

Simultaneity is not a fundamental property of events; it depends on the coordinate system. Is it possible to find a coordinate system in which any two events are simultaneous? The following example proves what was asserted in Section 12.6: there are two classes of events. For two given events, we can find either a coordinate system in which the events are simultaneous in time or one in which the events occur at the same point in space—but not both.

Example 12.7 The Order of Events: Timelike and Spacelike Intervals

Two events A and B on the x axis have the following coordinates in S :

Event A : (x_A, t_A) ; Event B : (x_B, t_B) .

The distance between the events is $L = x_B - x_A$ and the time T separating the events is $T = t_B - t_A$.

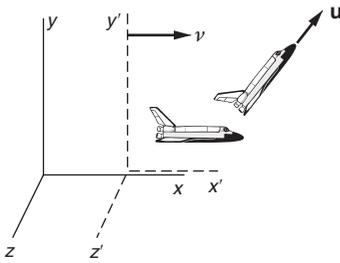
The distance between the events in the x' , y' system, as described by the Lorentz transformation, is

$$\begin{aligned} L' &= \gamma(L - vT). \\ T' &= \gamma \left(T - \frac{vL}{c^2} \right). \end{aligned}$$

Because v is always less than c , it follows that if $L > cT$ then L' is always positive, while T' can be positive, negative, or zero. Such an interval is called *spacelike*, since it is impossible to choose a system in which the events occur at the same place, though it is possible for them to be simultaneous, namely, in a system moving with $v = c^2T/L$. On the other hand, if $L < cT$, then T' is always positive and the events can never appear to be simultaneous, but L' can be positive, negative, or zero. The interval is then known as *timelike*, because it is impossible to find a coordinate system in which the events occur at the same time.

12.9 The Relativistic Addition of Velocities

The closest star is more than four light years away and our galaxy is roughly 10 000 light years across. Consequently, any method for traveling faster than light could be priceless for galactic exploration. Toward this goal, suppose we build a spaceship, the Starship Sophie, that can achieve a speed of $0.900c$. The crew of the Sophie then launches a second ship, Starship Surprise, that can reach $0.800c$. According to Newtonian rules, the Surprise should fly away at $1.700c$. Let's see what happens relativistically.



We designate our rest system (x, y, z, t) by S and spaceship Sophie's system (x', y', z', t') by S' . S' moves with velocity v along the x axis. The velocity of the Surprise, as observed from the Sophie, is $\mathbf{u}' = (u'_x, u'_y)$. Our task is to find the velocity \mathbf{u} of the Surprise that we observe in our rest system S .

From the definition of velocity, in S' we have

$$u'_x = \lim_{\Delta t' \rightarrow 0} \frac{\Delta x'}{\Delta t'}$$

$$u'_y = \lim_{\Delta t' \rightarrow 0} \frac{\Delta y'}{\Delta t'}$$

$$u'_z = \lim_{\Delta t' \rightarrow 0} \frac{\Delta z'}{\Delta t'}$$

The corresponding components in S are

$$u_x = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t}$$

$$u_y = \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t}$$

$$u_z = \lim_{\Delta t \rightarrow 0} \frac{\Delta z}{\Delta t}$$

The problem is to relate displacements and time intervals in S to those in S' . From the Lorentz transformation Eqs. (12.3) we have

$$\Delta x = \gamma(\Delta x' + v\Delta t')$$

$$\Delta y = \Delta y'$$

$$\Delta z = \Delta z'$$

$$\Delta t = \gamma(\Delta t' + (v/c^2)\Delta x')$$

Hence

$$\begin{aligned}\frac{\Delta x}{\Delta t} &= \frac{\gamma(\Delta x' + v\Delta t')}{\gamma[\Delta t' + (v/c^2)\Delta x']} \\ &= \frac{\Delta x'/\Delta t' + v}{1 + (v/c^2)(\Delta x'/\Delta t')}.\end{aligned}$$

Next we take the limit $\Delta t' \rightarrow 0$. Using $u'_x = \lim_{\Delta t' \rightarrow 0} \Delta x'/\Delta t'$, we obtain

$$u_x = \frac{u'_x + v}{1 + vu'_x/c^2}. \quad (12.8a)$$

Similarly,

$$u_y = \frac{u'_y}{\gamma[1 + vu'_x/c^2]} \quad (12.8b)$$

and

$$u_z = \frac{u'_z}{\gamma[1 + vu'_x/c^2]}. \quad (12.8c)$$

Equations (12.8) are the relativistic rules for adding velocities. For $v \ll c$, we obtain the Galilean result $\mathbf{u} = \mathbf{v} + \mathbf{u}'$.

The transformation from S to S' is

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2}. \quad (12.9a)$$

$$u'_y = \frac{u_y}{\gamma[1 - vu_x/c^2]}. \quad (12.9b)$$

$$u'_z = \frac{u_z}{\gamma[1 - vu_x/c^2]}. \quad (12.9c)$$

Returning to the problem of the two starships, let $u'_x = 0.800c$ be the speed of the Surprise relative to the Sophie and $v = 0.900c$ be the speed of the Sophie relative to us. The velocity of the Surprise relative to us is, from Eq. (12.8a),

$$\begin{aligned}u_x &= \frac{0.900c + 0.800c}{1 + (0.900)(0.800)} \\ &= \frac{1.700c}{1.720} = 0.988c.\end{aligned}$$

The speed of the Surprise is less than c . Equation (12.8a) reveals that we cannot exceed the speed of light by changing reference frames.

Taking the limiting case $u'_x = c$, the final velocity in the rest system is then

$$\begin{aligned}u_x &= \frac{c + v}{1 + vc/c^2} \\ &= c,\end{aligned}$$

independent of v . This agrees with the postulate that we built into the Lorentz transformation: the speed of light is the same for all observers. Furthermore, it suggests that the speed of light is the ultimate speed allowed by the theory of relativity.

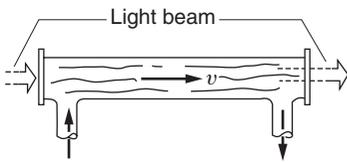
Example 12.8 The Speed of Light in a Moving Medium

As an exercise in the relativistic addition of velocities, let us find how a moving medium, such as flowing water, influences the speed of light.

The speed of light in matter is less than c . The index of refraction, n , is used to specify the speed in a medium:

$$n = \frac{c}{\text{velocity of light in the medium}}$$

$n = 1$ corresponds to empty space; in ordinary matter $n > 1$. The slowing can be appreciable: for water $n = 1.3$.



The problem is to find the speed of light through a moving liquid. For instance, consider a tube filled with water. If the water is at rest, the speed of light in the water with respect to the laboratory is $u = c/n$. What is the speed of light when the water is flowing with speed v ?

Consider the speed of light in water as observed in a coordinate system $S' = (x', y')$ moving with the water. The speed in S' is

$$u' = \frac{c}{n}$$

The speed in the laboratory is, by Eq. (12.8a),

$$\begin{aligned} u &= \frac{u' + v}{1 + u'v/c^2} \\ &= \frac{c/n + v}{1 + v/nc} \\ &= \frac{c}{n} \left(\frac{1 + nv/c}{1 + v/nc} \right). \end{aligned}$$

If we expand the factor on the right-hand side and neglect terms of order $(v/c)^2$ and smaller, we obtain

$$\begin{aligned} u &= \frac{c}{n} \left(1 + \frac{nv}{c} - \frac{v}{nc} \right) \\ &= \frac{c}{n} + v \left(1 - \frac{1}{n^2} \right). \end{aligned}$$

The light appears to be “dragged” by the fluid, but not completely. Only the fraction $f = 1 - 1/n^2$ of the fluid velocity is added to the speed of light c/n . This effect was observed experimentally in 1851 by Fizeau, although it was not explained satisfactorily until the advent of relativity.

12.10 The Doppler Effect

The Doppler effect is the change in the frequency of a wave due to motion between the source and observer. It causes the familiar drop in pitch of the horn of a truck or the whistle of a train as they pass by. For astronomers and astrophysicists the Doppler effect provides an invaluable tool for measuring the speed of far off objects by the shift in the spectral wavelengths they emit. All our knowledge about how fast the universe is expanding comes from observations of the Doppler effect in spectral lines. More prosaically, the Doppler effect is at the heart of reliable and cheap radar speed monitors.

The relativistic Doppler effect differs from the classical effect in a pleasing manner: it is simpler. Furthermore, it displays a phenomenon absent in classical behavior, the transverse Doppler effect that causes a frequency shift in light from a moving source, as seen by an observer transverse to the path.

To start, we review the classical Doppler effect in sound.

12.10.1 The Doppler Effect in Sound

Sound travels through a medium, such as air, with a speed w determined by the properties of the medium, independent of the motion of the source.

Consider sound waves from a source moving with velocity w through the medium toward an observer at rest. For now, we shall restrict ourselves to the case where the observer is along the line of motion. We will picture sound as regular series of pulses separated by time $\tau_0 = 1/\nu_0$, where ν_0 is the number of pulses per second generated by the source. The distance between pulses is $w\tau_0 = w/\nu_0$, which we designate by λ . We could equally well picture the disturbance as a sine wave, in which case ν_0 corresponds to the frequency of sound and the distance between successive crests is the wavelength $\lambda = w/\nu_0$.

If the source moves toward the observer at speed v , then the distance between successive pulses is $\lambda_D = \lambda - v\tau_0 = \lambda - v/\nu_0$. Hence

$$\frac{w}{\nu'_0} = \frac{w}{\nu_0} - \frac{v}{\nu_0},$$

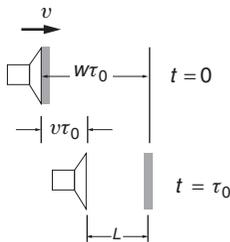
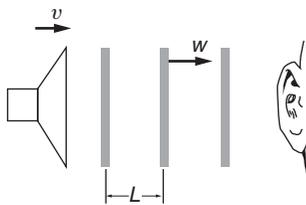
$$\nu'_0 = \nu_0 \left(\frac{1}{1 - v/w} \right) \quad (\text{moving source}). \quad (12.10)$$

The shift in frequency $(\Delta\nu)\lambda_D = \nu'_0 - \nu_0$ is known as the *Doppler shift*.

The situation is somewhat different if the observer is moving toward the source at speed w . Previously, the observer was at rest in the medium: now the observer is moving through the medium. The relative velocity between source and observer is the same, w . The distance between wave fronts is unchanged, but the relative speed of arrival is now $w + v$. Consequently, the frequency is $\nu'_0 = (w + v)/\lambda$, which can be written

$$\nu'_0 = \nu_0(1 + v/w) \quad (\text{moving observer}). \quad (12.11)$$

Equations (12.11) and (12.12) are identical to first order in the ratio v/w , but they differ in the second order. The second-order difference could



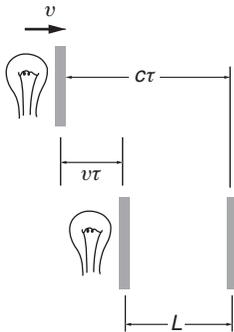
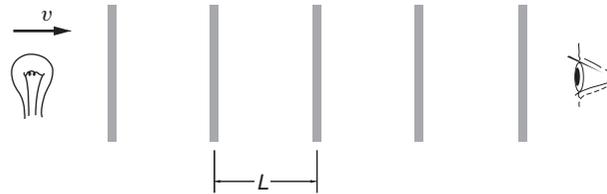
in principle be used to determine whether the Doppler shift is due to motion of the source or motion of the observer. The distinction is real because the motion is measured relative to a fixed medium such as air.

If these results were valid for light waves in space, we would be able to distinguish which of two systems is at absolute rest, which is not possible. To resolve this difficulty, we turn now to a relativistic derivation of the Doppler effect.

12.10.2 The Relativistic Doppler Effect

A light source flashes with period $\tau_0 = 1/\nu_0$ in its rest frame. The source is moving toward an observer with velocity v . Due to time dilation, the period in the observer's rest frame is

$$\tau = \gamma\tau_0.$$



If the wavelength λ_D is the distance between pulses in the observer's rest frame, the frequency of the pulses is $\nu_D = c/\lambda_D$, where the wavelength λ_D is the distance between pulses in the observer's frame. Because the source is moving toward the observer this distance is

$$\lambda_D = c\tau - v\tau = (c - v)\tau$$

and

$$\begin{aligned} \nu_D &= \frac{c}{(c - v)\tau} \\ &= \left(\frac{1}{1 - v/c} \right) \frac{1}{\gamma\tau_0} \end{aligned}$$

or

$$\nu_D = \nu_0 \frac{\sqrt{1 - v^2/c^2}}{1 - v/c}$$

which reduces to

$$\nu_D = \nu_0 \sqrt{\frac{1 + v/c}{1 - v/c}}. \tag{12.12}$$

ν_D is the frequency in the observer's rest frame and v is the relative speed of source and observer. As we expect, there is no mention of motion relative to a medium. The relativistic result plays no favorites with the classical results; it disagrees with both Eqs. (12.10) and (12.11) but treats

the case of moving source and moving observer symmetrically: it is the geometric mean of the two classical results.

12.10.3 The Doppler Effect Off the Line of Motion

We have analyzed the Doppler effect when the source and observer move along the line connecting them but this is not the most general situation. For instance, consider a satellite broadcasting a radio beacon signal to a ground tracking station that monitors the Doppler-shifted frequency.

We can readily generalize our method to find the Doppler effect for an observer in a direction at angle θ from the line of motion. We again visualize the source as a flashing light. The period of the flashes in the observer's rest frame is $\tau = \gamma\tau_0$, as before. The frequency seen by the observer is c/λ_D . The source moves a distance $v\tau$ between flashes and it is apparent from the sketch that

$$\begin{aligned}\lambda_D &= c\tau - v\tau \cos \theta \\ &= (c - v \cos \theta)\tau.\end{aligned}$$

Hence

$$\begin{aligned}v_D &= \frac{c}{\lambda_D} \\ &= \frac{c}{(c - v \cos \theta)\tau_0\gamma} \\ v_D &= v_0 \frac{\sqrt{1 - v^2/c^2}}{1 - (v/c) \cos \theta}\end{aligned}\tag{12.13}$$

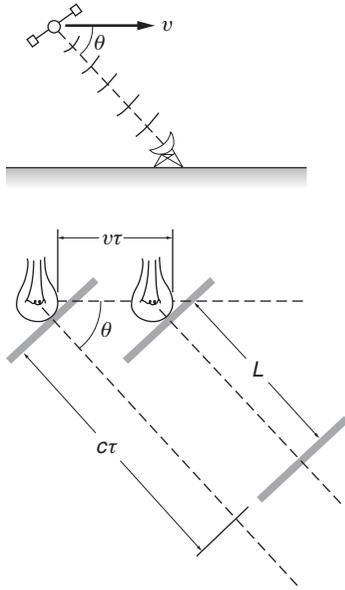
where we have used $\tau_0 = 1/\nu_0$. In this result, θ is the angle measured in the rest frame of the observer. Along the line of motion, $\theta = 0$ and we recover our previous result, Eq. (12.12). At $\theta = \pi/2$ the relative velocity between source and observer is zero. The classical Doppler effect would vanish here, but relativistically there is a shift in frequency; ν_D differs from ν_0 by the factor $\sqrt{1 - v^2/c^2}$. This “transverse” Doppler effect is due to time dilation. The flashing lamp is effectively a moving clock and moving clocks run slow.

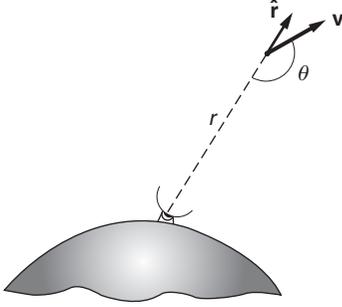
The relativistic Doppler effect agrees with the classical result to order v/c , so that any experiment to differentiate between them must be sensitive to effects of order $(v/c)^2$. Nevertheless, the relativistic expression was confirmed by H.E. Ives and G.R. Stilwell in 1938 by observing small shifts in the wavelengths emitted by fast-moving atoms.

A useful application of the Doppler effect is in navigational systems, as the following example explains.

Example 12.9 Doppler Navigation

The Doppler effect can be used to track a moving body, such as a satellite, from a reference point on the Earth. This provided the basis for a navigational system that was created when the first satellites were





flown. Although it has been superseded by the Global Positioning System (GPS), the method is remarkably accurate; changes in the position of a satellite 10^8 m away can be determined to a fraction of a centimeter.

Consider a satellite moving with velocity \mathbf{v} at some distance r from a ground station. An oscillator on the satellite broadcasts a signal with frequency ν_0 . Since $v \ll c$ for satellites, we can approximate Eq. (12.13) by retaining only terms of order v/c . The frequency ν_D received by the ground station can then be written

$$\begin{aligned} \nu_D &\approx \frac{\nu_0}{1 - (v/c) \cos \theta} \\ &\approx \nu_0 \left(1 + \frac{v}{c} \cos \theta \right). \end{aligned}$$

There is an oscillator in the ground station identical to the one in the satellite. At rest, both oscillators run at the same frequency ν_0 with corresponding wavelength $\lambda_0 = c/\nu_0$. In flight, the observed satellite frequency is different, and by simple electronic methods the difference frequency (“beat” frequency) $\nu_D - \nu_0$ can be measured:

$$\nu_D - \nu_0 = \nu_0 \frac{v}{c} \cos \theta.$$

The radial velocity of the satellite is

$$\begin{aligned} \frac{dr}{dt} &= \hat{\mathbf{r}} \cdot \mathbf{v} \\ &= -v \cos \theta. \end{aligned}$$

Hence

$$\begin{aligned} \frac{dr}{dt} &= -\frac{c}{\nu_0} (\nu_D - \nu_0) \\ &= -\lambda_0 (\nu_D - \nu_0). \end{aligned}$$

ν_D varies in time as the satellite’s velocity and direction change. To find the total radial distance traveled between times T_a and T_b , we integrate the above expression with respect to time:

$$\begin{aligned} \int_{T_a}^{T_b} \left(\frac{dr}{dt} \right) dt &= -\lambda_0 \int_{T_a}^{T_b} (\nu_D - \nu_0) dt \\ r_b - r_a &= -\lambda_0 \int_{T_a}^{T_b} (\nu_D - \nu_0) dt. \end{aligned}$$

The integral is the number of cycles N_{ba} of the beat frequency that occur in the interval T_a to T_b . (One cycle occurs in a time $\tau = 1/(\nu_D - \nu_0)$, so that $\int dt/\tau$ is the total number of cycles.) Hence

$$r_b - r_a = -\lambda_0 N_{ba}.$$

This result has a simple interpretation: whenever the radial distance increases by one wavelength, the phase of the beat signal decreases one cycle. Similarly, when the radial distance decreases by one wavelength, the phase of the beat signal increases by one cycle.

Satellite communication systems operate at a typical wavelength of 10 cm, and since the beat signal can be measured to a fraction of a cycle, satellites can be tracked to about 1 cm. If the satellite and ground-based oscillators do not each stay tuned to the same frequency, ν_0 , there will be an error in the beat frequency. To avoid this problem a two-way Doppler tracking system can be used in which a signal from the ground is broadcast to the satellite, which then amplifies it and relays it back to the ground. This has the added advantage of doubling the Doppler shift, increasing the resolution by a factor of 2.

We sketched the principles of Doppler navigation for the classical case $v \ll c$. For certain tracking applications the precision is so high that relativistic effects must be taken into account.

As we have shown, a Doppler tracking system also gives the instantaneous radial velocity of the satellite $v_r = -c(\nu_D - \nu_0)/\nu_0$. This is particularly handy, since both velocity and position are needed to check satellite trajectories. A more prosaic use of this result is in police radar speed monitors: a microwave signal is reflected from an oncoming car and the beat frequency of the reflected signal reveals the car's speed.

12.11 The Twin Paradox

Among the paradoxes that add to the fascination of special relativity, probably none has generated more discussion than the twin paradox. The paradox is simple to state: two twins, Alice and Bob, have identical clocks. Alice sets out on a long space voyage while Bob remains at home. Suppose that the spacecraft flies away in a straight line at constant speed v for time $T_0/2$ as measured by Alice using her onboard clock. She quickly reverses speed and heads back, returning home at T_0 . Alice would observe that she has aged by time T_0 , as measured with her onboard clock.

Because of time dilation, Bob would observe that the time for the journey is

$$T'_B = \gamma T_0 \approx T_0 \left(1 + \frac{1}{2} \frac{v^2}{c^2} \right).$$

Consequently, Bob concludes that because of the time dilation he is older than Alice by

$$\Delta T_{A,B} = \frac{1}{2} T_0 \frac{v^2}{c^2}, \quad (12.14)$$

or, equivalently, that Alice is younger than he.

If this same argument were applied by Alice, she would conclude that she is older than Bob by

$$\Delta T_{B,A} = \frac{1}{2} T_0 \frac{v^2}{c^2} \quad (12.15)$$

or that Bob is correspondingly younger than she.

Obviously they cannot both be right. Who is younger? Is there really any difference?

The paradox arises from ignoring the fact that the situations for the twins are not equivalent. Bob's system is inertial but for part of the time, Alice's is not. She must reverse her velocity in order to return to the starting point and while her velocity is changing, her system is not inertial. During this interval the situation becomes asymmetric: there is no question as to which twin is accelerating. If each were carrying an accelerometer such as a mass on a spring, Bob's would remain at zero while Alice's would show a large deflection as the spaceship reversed direction.

In principle, analyzing events in an accelerating system requires general relativity. Nevertheless, we can find the leading terms of the solution by invoking the equivalence principle and the analysis of the gravitational clock shift in Chapter 9. Recall that according to the principle of equivalence, there is no way to distinguish between an acceleration a and a uniform gravitational field $g = -a$. Due to the gravitational clock shift, Alice sees Bob's clock speed up during turnaround. We shall see that this time advance brings the two observers into agreement.

During turnaround, let us suppose that Alice experiences a uniform acceleration a applied for time τ_t . The time required to reverse the velocity is $a\tau_t = 2v$. During this time Alice experiences an effective gravitational field $g_{\text{eff}} = -a$ that points from Bob to her. As a result, Alice sees that Bob's clock has sped up due to the gravitational red shift (in this case, actually a blue shift). The fractional shift in the rate of the clock is $g_{\text{eff}}h/c^2$, where h is the "height" of the clock in the gravitational field. Turnaround occurs at time $T_0/2$, so $h = vT_0/2$. The total advance Alice measures in Bob's clock during turnaround is

$$\Delta T_{\text{grav}} = \frac{g_{\text{eff}}h}{c^2\tau}.$$

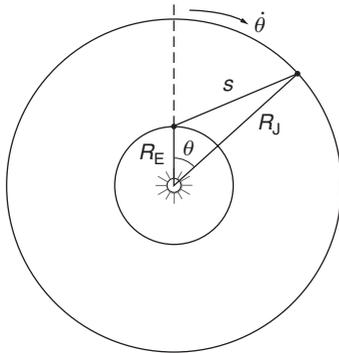
Inserting the values $h = vT_0/2$, $g_{\text{eff}} = a = 2v/\tau$, and $\tau = v/a$, we have

$$\Delta T_{\text{grav}} = \frac{av^2T_0}{c^2a} = T_0 \frac{v^2}{c^2}. \quad (12.16)$$

Before taking the gravitational frequency shift into account, Alice believed that Bob was younger than she by $(1/2)T_0(v^2/c^2)$. However, when ΔT_{grav} is added to this time, she realizes that Bob is actually older by that amount. Both twins agree: at the end of the trip Alice is younger than Bob by $T_0v^2/2c^2$. It appears that travel helps one stay relatively youthful.

Problems

For problems marked *, refer to page 524 for a hint, clue, or answer.



12.1 Maxwell's proposal*

Section 12.3 mentioned Maxwell's proposal for measuring the effect of source motion on the speed of light, using Jupiter's moons as clocks. In the sketch (not to scale), the inner circle is the Earth's orbit and the outer circle is Jupiter's orbit. The angle θ is the position of Jupiter with respect to the Earth's position. Jupiter's period is 11.9 years and the Earth's period is 1 year, so that $\dot{\theta} = 2\pi(11.9 - 1) \text{ rad/year} = 2.2 \times 10^{-6} \text{ rad/s}$. The radius of Jupiter's orbit is $R_J = 7.8 \times 10^{11} \text{ m}$ and the radius of Earth's orbit is $R_E = 1.5 \times 10^{11} \text{ m}$.

The problem is to find the time delay ΔT predicted by Maxwell's method. If s is the distance between Jupiter and Earth then

$$\Delta T = \frac{s}{(c - \dot{s})} - \frac{s}{(c + \dot{s})} \approx \frac{2s\dot{s}}{c^2}.$$

Calculate the maximum value of ΔT .

12.2 Refined Michelson–Morley interferometer

The improved apparatus used in 1887 by Michelson and Morley at the Case School of Applied Science (now Case-Western Reserve University) could detect a 0.01 fringe using sodium light, $\lambda = 590 \text{ nm}$.

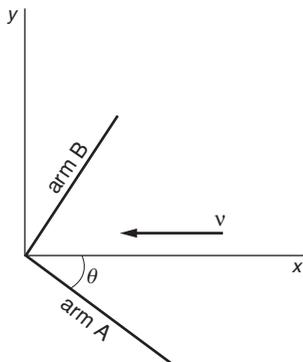
What is the upper limit to the Earth's velocity with respect to the ether set by this experiment? For comparison, the Earth's orbital velocity around the Sun is 30 km/s.

12.3 Skewed Michelson–Morley apparatus

In Section 12.3 arm A of the Michelson–Morley interferometer was assumed to be along the line of motion and arm B perpendicular, and the predicted time difference according to the ether theory was

$$\Delta\tau = \frac{l}{c} \left(\frac{v^2}{c^2} \right).$$

Calculate the expected time difference if arm A is at angle θ to the line of motion through the ether, as shown.



12.4 Asymmetric Michelson–Morley interferometer

If the two arms of the Michelson interferometer have different lengths l_1 and l_2 , show that the fringe shift when the interferometer is rotated by 90° with respect to the velocity v through the ether is

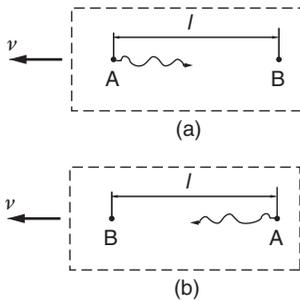
$$N = \left(\frac{l_1 + l_2}{\lambda} \right) (v^2/c^2)$$

where λ is the wavelength of the source light.

12.5 *Lorentz–FitzGerald contraction*

The Irish physicist G.F. FitzGerald and the Dutch physicist H.A. Lorentz tried to account for the null result of the Michelson–Morley experiment by the conjecture that movement through the ether sets up a strain that causes contraction along the line of motion by the factor $1 - \frac{1}{2}v^2/c^2$.

Show that this hypothesis can account for the absence of fringe shift in the Michelson–Morley experiment. (The hypothesis was disproved in 1932 by experimenters who used an interferometer with unequal arms.)



12.6 *One-way test of the constancy of c*

Light in a Michelson–Morley interferometer makes a roundtrip, and the predicted time delay is second order, proportional to v^2/c^2 .

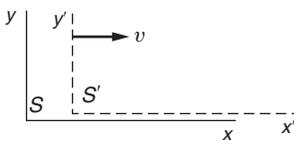
Here is an experiment that would give a first-order result proportional to v/c . Consider a laboratory moving through the ether with speed v in the direction shown. The observers have clocks and light pulsers. At time $t = 0$ A sends a signal to B a distance l away, sketch (a). B records the arrival time. The laboratory is then rotated 180° , reversing the positions of A and B. At time $t = T$, A sends a second signal to B, sketch (b).

(a) Show that according to the ether theory, the interval that B observes between the signals is $T + \Delta T$, where

$$\Delta T \approx \frac{2lv}{c^2}$$

correct to order $(v/c)^3$.

(b) Assume that one clock in this experiment is on the ground and the other is in a satellite overhead. For a circular orbit with a period of 24 hours, $l = 5.6 R_e$, where R_e is the Earth’s radius = 6.4×10^6 m. Using an atomic clock stable to within 1 part in 10^{16} , what is the smallest value of v this experiment could detect?



12.7 *Four events*

Note: S refers to an inertial system x, y, z, t and S' refers to an inertial system x', y', z', t' , moving along the x axis with speed v relative to S . The origins coincide at $t = t' = 0$. For numerical work, take $c = 3 \times 10^8$ m/s.

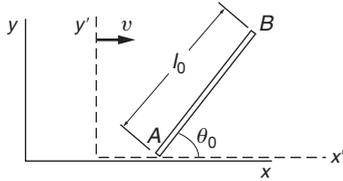
Assuming that $v = 0.6c$, find the coordinates in S' of the following events

- (a) $x = 4$ m, $t = 0$ s.
- (b) $x = 4$ m, $t = 1$ s.
- (c) $x = 1.8 \times 10^8$ m, $t = 1$ s.
- (d) $x = 10^9$ m, $t = 2$ s.

12.8 *Relative velocity of S and S'*

Refer to the note and the sketch in Problem 12.7.

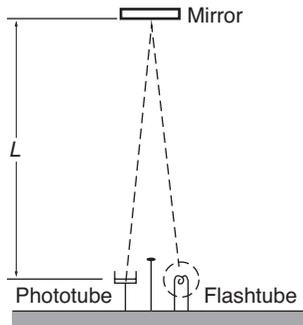
An event occurs in S at $x = 6 \times 10^8$ m, and in S' at $x' = 6 \times 10^8$ m, $t' = 4$ s. Find the relative velocity of the systems.

12.9 *Rotated rod*

A rod of length l_0 lies in the $x'y'$ plane of its rest system and makes an angle θ_0 with the x' axis. What is the length and orientation of the rod in the lab system x, y in which the rod moves to the right with velocity v ?

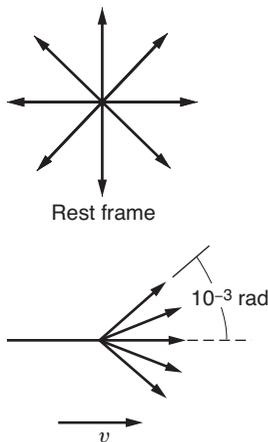
12.10 *Relative speed**

An observer sees two spaceships flying apart with speed $0.99c$. What is the speed of one spaceship as viewed by the other?

12.11 *Time dilation*

The clock in the sketch can provide an intuitive explanation of the time dilation formula. The clock consists of a flashtube, mirror, and phototube. The flashtube emits a pulse of light that travels distance L to the mirror and is reflected back to the phototube. Every time a pulse hits the phototube it triggers the flashtube. Neglecting time delay in the triggering circuits, the period of the clock is $\tau_0 = 2L/c$.

Now examine the clock in a coordinate system moving to the left with uniform velocity v . In this system the clock appears to move to the right with velocity v . Find the period of the clock in the moving system by direct calculation, using only the assumptions that c is a universal constant, and that distance perpendicular to the line of motion is unaffected by the motion. The result should be identical to that given by the Lorentz transformation: $\tau = \tau_0 / \sqrt{1 - v^2/c^2}$.

12.12 *Headlight effect**

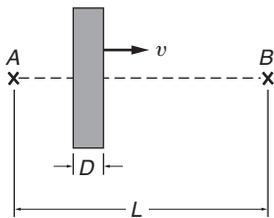
A light beam is emitted at angle θ_0 with respect to the x' axis in S' .

(a) Find the angle θ the beam makes with respect to the x axis in S .

(b) A source that radiates light uniformly in all directions in its rest frame radiates strongly in the forward direction in a frame in which it is moving with speed v close to c . This is called the headlight effect; it is very pronounced in synchrotron light sources in which electrons moving at relativistic speeds emit light in a narrow cone in the forward direction. Using the result of part (a), find the speed of a source for which half the radiation is emitted in a cone subtending 10^{-3} rad. (The sketch is considerably exaggerated, because 10^{-3} rad is only about 0.06 degree.)

12.13 *Moving mirror*

The frequency of light reflected from a moving mirror undergoes a Doppler shift because of the motion of the image. Find the Doppler shift of light reflected directly back from a mirror that is approaching the observer with speed v , and show that it is the same as if the image were moving toward the observer at speed $2v/(1 + v^2/c^2)$.



12.14 *Moving glass slab**

A slab of glass moves to the right with speed v . A flash of light is emitted from A and passes through the glass to arrive at B a distance L away. The glass has thickness D in its rest frame, and the speed of light in the glass is c/n . How long does it take the light to go from A to B ?

12.15 *Doppler shift of a hydrogen spectral line**

One of the most prominent spectral lines of hydrogen is the H_α line, a bright red line with a wavelength of 656.1×10^{-9} m.

(a) What is the expected wavelength of the H_α line from a star receding with a speed of 3000 km/s?

(b) The H_α line measured on Earth from opposite ends of the Sun's equator differ in wavelength by 9×10^{-12} m. Assuming that the effect is caused by rotation of the Sun, find the period of rotation. The diameter of the Sun is 1.4×10^6 km.

12.16 *Pole-vaulter paradox**

The pole-vaulter has a pole of length l_0 , and the farmer has a barn $\frac{3}{4}l_0$ long. The farmer bets that he can shut the front and rear doors of the barn with the pole completely inside. The bet being made, the farmer asks the pole-vaulter to run into the barn with a speed $v = c\sqrt{3}/2$. In this case the farmer observes the pole to be Lorentz contracted to $l = l_0/2$, and the pole fits into the barn with ease. The farmer slams the door the instant the pole is inside, and claims the bet. The pole-vaulter disagrees: he sees the barn contracted by a factor of 2, so the pole can't possibly fit inside. Let the farmer and barn be in system S and the pole-vaulter in system S' . Call the leading end of the pole A , and the trailing end B .



(a) The farmer in S sees A reach the rear door at $t_A = 0$, and closes the front door at the same time $t_A = t_B = 0$. What is the length of the pole as seen in S ?

(b) The pole-vaulter in S' sees A reach the rear door at t'_A . Where does the pole-vaulter see B at this instant?

(c) Show that in S' , A and B do not lie inside the barn at the same instant.

12.17 Transformation of acceleration

The relativistic transformation of acceleration from S' to S can be found by extending the procedure of Section 12.9. The most useful transformation is for the case in which the particle is instantaneously at rest in S' but is accelerating at rate a_0 parallel to the x' axis.

Show that for this case the x acceleration in S is given by $a_x = a_0/\gamma^3$.

12.18 The consequences of endless acceleration*

The relativistic transformation for acceleration derived in Problem 12.17 shows the impossibility of accelerating a system to a velocity greater than c . Consider a spaceship that accelerates at constant rate a_0 as measured by an accelerometer carried aboard, for instance a mass stretching a spring.

(a) Find the speed after time t for an observer in the system in which the spaceship was originally at rest.

(b) The speed predicted classically is $v_0 = a_0 t$. What is the actual speed for the following cases: $v_0 = 10^{-3}c$, c , 10^3c .

12.19 Traveling twin

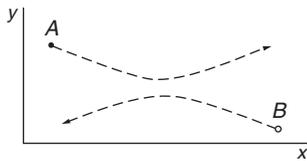
A young man voyages to the nearest star, α Centauri, 4.3 light years away. He travels in a spaceship at a velocity of $c/5$. When he returns to Earth, how much younger is he than his twin brother who stayed home?

13 RELATIVISTIC DYNAMICS

- 13.1 Introduction 478
- 13.2 Relativistic Momentum 478
- 13.3 Relativistic Energy 481
- 13.4 How Relativistic Energy and Momentum are Related 487
- 13.5 The Photon: A Massless Particle 488
- 13.6 How Einstein Derived $E = mc^2$ 498
- Problems* 499

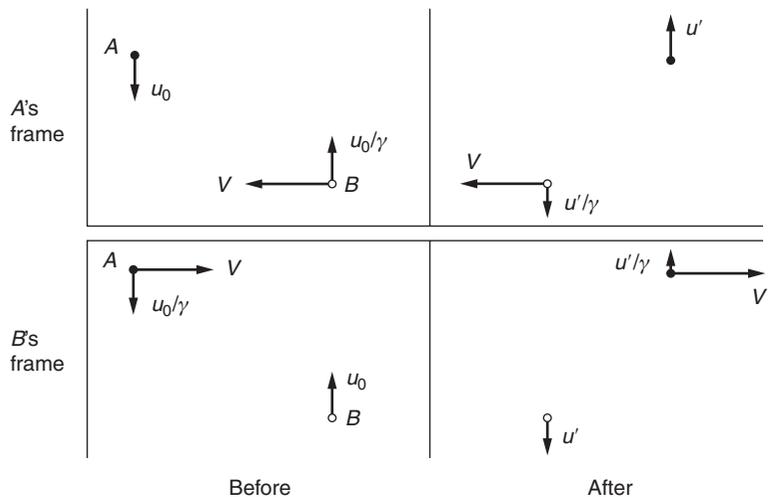
13.1 Introduction

In Chapter 12 we saw how the postulates of special relativity lead to new kinematical relations for space and time. These relations can naturally be expected to have important implications for dynamics, particularly for the meaning of momentum and energy. In this chapter we examine the modifications to the Newtonian concepts of momentum and energy required by special relativity. The underlying strategy is to ensure that momentum and energy in an isolated system continue to be conserved. This approach is often used in extending the frontiers of physics: by reformulating conservation laws so that they are preserved in new situations, we are led to generalizations of familiar concepts. We can also be led to the discovery of unfamiliar concepts, for instance the concept of massless particles that can nevertheless carry energy and momentum.



13.2 Relativistic Momentum

To investigate the nature of momentum in special relativity, consider a glancing elastic collision between two identical particles A and B in an isolated system. We want the total momentum of the system to be conserved, as it is in non-relativistic physics. We shall view the collision in two frames: A 's frame, the frame moving along the x axis with A so that A is at rest while B approaches along the x direction with speed V , and then in B 's frame, which is moving with B in the opposite direction so that B is at rest and A is approaching. (The term “frame” is used synonymously with “reference system.”) We take the collisions to be completely symmetrical. Each particle has the same y speed u_0 in its own frame before the collision, as shown in the sketches. The effect of the collision is to reverse the y velocities but leave the x velocities unchanged.



The relative x velocity of the frames is V . In A 's frame, the y velocity of particle A is u_0 , and by the transformation of velocities, Eqs. (12.8)

and (12.9), the y velocity of particle B is u_0/γ where $\gamma = 1/\sqrt{1 - V^2/c^2}$. The situation is symmetrical when viewed from B 's frame.

After the collision the y velocities have reversed their directions as shown. The situation remains symmetric: if the y velocity of A or B in its own frame is u' , the y velocity of the other particle is u'/γ .

Our task is to find a conserved quantity analogous to classical momentum. We suppose that the momentum of a particle moving with velocity \mathbf{w} is

$$\mathbf{p} = m(w)\mathbf{w},$$

where $m(w)$ is a scalar quantity, yet to be determined, analogous to Newtonian mass but which could depend on the speed w .

The x momentum in A 's frame is due entirely to particle B . Before the collision B 's speed is $w = \sqrt{V^2 + u_0^2/\gamma^2}$ and after the collision it is $w' = \sqrt{V^2 + u'^2/\gamma^2}$. Imposing conservation of momentum in the x direction yields

$$m(w)V = m(w')V.$$

It follows that $w = w'$, so that

$$u' = u_0.$$

In other words, y motion is reversed in the A frame.

Next we write the statement of the conservation of momentum in the y direction as evaluated in A 's frame. Equating the y momentum before and after the collision gives

$$-m_0u_0 + m(w)\frac{u_0}{\gamma} = m_0u_0 - m(w)\frac{u_0}{\gamma}$$

which gives

$$m(w) = \gamma m_0.$$

In the limit $u_0 \rightarrow 0$, $m(u_0) \rightarrow m(0)$, which we take to be the Newtonian mass, or "rest mass" m_0 , of the particle. In this limit, $w = V$. Hence

$$m(V) = \gamma m_0 = \frac{m_0}{\sqrt{1 - V^2/c^2}}. \quad (13.1)$$

Consequently, momentum is preserved in the collision provided we define the momentum of a particle moving with velocity \mathbf{v} to be

$$\mathbf{p} = m\mathbf{v} \quad (13.2)$$

where

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = \gamma m_0.$$

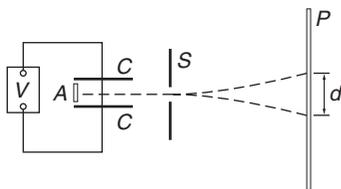
The quantity $m = \gamma m_0$ is referred to as the "relativistic mass" or more often simply as the mass of a particle. If the rest mass is intended, that needs to be made specific.

In relativity there is an upper limit to speed: the speed of light. However, there is no upper limit on momentum. Once a particle is moving with speed close to c , an increase in momentum comes about primarily through an increase in mass. High energy particle accelerators do not make particles go substantially faster and faster. A particle is quickly accelerated to speed close to c . After that, the accelerator principally makes the particle more and more massive with only a very small increment in speed.

The expression $\mathbf{p} = m\mathbf{v} = \gamma m_0\mathbf{v}$ is sometimes taken as the starting point for developing relativistic dynamics, but in the early days of relativity attention was focused not so much on momentum but on the apparent dependence of mass on speed. Investigation of this problem provided the first direct experimental evidence for Einstein's theory.

Example 13.1 Speed Dependence of the Electron's Mass

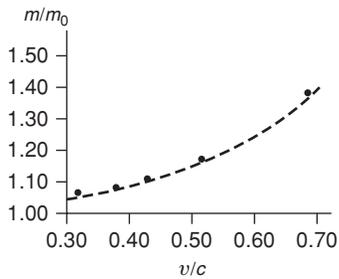
At the beginning of the twentieth century there were several speculative theories based on various models of the structure of the electron that predicted that the mass of an electron would vary with its speed. One theory, from Max Abraham (1902), predicted $m = m(u_0)[1 + \frac{2}{5}(v^2/c^2)]$ for $v \ll c$ and another from Hendrik A. Lorentz (1904) gave $m = m_0/\sqrt{1 - v^2/c^2} \approx m(u_0)[1 + \frac{1}{2}(v^2/c^2)]$. The Abraham theory, which retained the idea of the ether drift and absolute motion, predicted no time dilation effect. Lorentz's result, while identical in form to that published by Einstein in 1905, was derived using the ad hoc Lorentz contraction and did not possess the generality of Einstein's theory. Experimental work on the effect of speed on the electron's mass was initiated by Kaufmann in Göttingen in 1902. His data favored the theory of Abraham, and in a 1906 paper he rejected the Lorentz–Einstein results. However, further work by Bestelmeyer (1907) in Göttingen and Bucherer (1909) in Bonn revealed errors in Kaufmann's work and confirmed the Lorentz–Einstein formula.



Physicists were in agreement that the force on a moving electron in an applied electric field \mathbf{E} and magnetic field \mathbf{B} is $q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ (the units are SI), where q is the electron's charge and \mathbf{v} its velocity. Bucherer employed this force law in the apparatus shown in the sketch. The apparatus is evacuated and immersed in an external magnetic field \mathbf{B} perpendicular to the plane of the sketch. The source of the electrons A is a button of radioactive material, generally radium salts. The emitted electrons ("beta-rays") have a broad energy spectrum extending to 1 MeV or so. To select a single speed, the electrons are passed through a "velocity filter" composed of a transverse electric field \mathbf{E} (produced between two parallel metal plates C by the battery V) and the perpendicular magnetic field. \mathbf{E} , \mathbf{B} , and \mathbf{v} are mutually perpendicular. The transverse force is zero when $qE = qvB$, so that electrons with $v = E/B$ are undeflected and are able to pass through the slit S .

Beyond S only the magnetic field acts. The electrons move with constant speed v and are bent into a circular path by the magnetic force $q\mathbf{v}\mathbf{B}$. The radius of curvature R is given by $mv^2/R = qvB$, or $R = mv/qB = (m/q)(E/B^2)$.

The electrons eventually strike the photographic plate P , leaving a trace. By reversing \mathbf{E} and \mathbf{B} , the sense of deflection is reversed. R is found from a measurement of the total deflection d and the known geometry of the apparatus. E and B are measured by standard techniques. Finding R for different velocities allowed the velocity dependence of m/q to be measured. Physicists believe that charge does not vary with velocity (otherwise an atom would not stay strictly neutral in spite of how the energy of its electrons varied), so that the variation of m/q can be attributed to variation in m alone.



The graph shows Bucherer's data together with a dashed line corresponding to the Einstein prediction $m = m_0 / \sqrt{1 - v^2/c^2}$. The agreement is striking.

Today, the relativistic equations of motion are used routinely to design high energy particle accelerators. For protons, accelerators have been operated with m/m_0 up to 10^4 , while for electrons the ratio $m/m_0 = 10^5$ has been reached. The successful operation of these machines leaves no doubt of the validity of relativistic dynamics.

13.3 Relativistic Energy

By generalizing the Newtonian concept of energy, we can find a corresponding relativistic quantity that is also conserved. Recall the argument from Chapter 5: the change in kinetic energy K of a particle as it moves from \mathbf{r}_a to \mathbf{r}_b under the influence of force \mathbf{F} is

$$\begin{aligned} K_b - K_a &= \int_a^b \mathbf{F} \cdot d\mathbf{r} \\ &= \int_a^b \frac{d\mathbf{p}}{dt} \cdot d\mathbf{r}. \end{aligned}$$

For a Newtonian particle moving with velocity \mathbf{u} the momentum is given by $\mathbf{p} = m\mathbf{u}$, where m is constant. Then

$$\begin{aligned} K_b - K_a &= \int_a^b \frac{d}{dt}(m\mathbf{u}) \cdot d\mathbf{r} \\ &= \int_a^b m \frac{d\mathbf{u}}{dt} \cdot \mathbf{u} dt \\ &= \int_a^b m\mathbf{u} \cdot d\mathbf{u}. \end{aligned}$$

Using the identity $\mathbf{u} \cdot d\mathbf{u} = \frac{1}{2}d(\mathbf{u} \cdot \mathbf{u}) = \frac{1}{2}d(u^2) = u du$, we obtain

$$K_b - K_a = \frac{1}{2}mu_b^2 - \frac{1}{2}mu_a^2.$$

It is natural to try the same procedure starting with the relativistic expression for momentum $\mathbf{p} = m_0\mathbf{u}/\sqrt{1-u^2/c^2}$:

$$\begin{aligned} K_b - K_a &= \int_a^b \frac{d\mathbf{p}}{dt} \cdot d\mathbf{r} \\ &= \int_a^b \frac{d\mathbf{p}}{dt} \cdot \frac{d\mathbf{r}}{dt} dt \\ &= \int_a^b \frac{d}{dt} \left[\frac{m_0\mathbf{u}}{\sqrt{1-u^2/c^2}} \right] \cdot \mathbf{u} dt \\ &= \int_a^b \mathbf{u} \cdot d \left[\frac{m_0\mathbf{u}}{\sqrt{1-u^2/c^2}} \right]. \end{aligned}$$

The integrand has the form $\mathbf{u} \cdot d\mathbf{p}$. Using the relation $\mathbf{u} \cdot d\mathbf{p} = d(\mathbf{u} \cdot \mathbf{p}) - \mathbf{p} \cdot d\mathbf{u}$ gives

$$\begin{aligned} K_b - K_a &= (\mathbf{u} \cdot \mathbf{p}) \Big|_a^b - \int_a^b \mathbf{p} \cdot d\mathbf{u} \\ &= \frac{m_0u^2}{\sqrt{1-u^2/c^2}} \Big|_a^b - \int_a^b \frac{m_0u du}{\sqrt{1-u^2/c^2}}, \end{aligned}$$

where we have again used the identity $\mathbf{u} \cdot d\mathbf{u} = u du$. The integral is elementary, and we find

$$K_b - K_a = \frac{m_0u^2}{\sqrt{1-u^2/c^2}} \Big|_a^b + m_0c^2 \sqrt{1-\frac{u^2}{c^2}} \Big|_a^b.$$

Let point b be arbitrary and take the particle to be at rest at point a so $u_a = 0$:

$$\begin{aligned} K &= \frac{m_0u^2}{\sqrt{1-u^2/c^2}} + m_0c^2 \sqrt{1-\frac{u^2}{c^2}} - m_0c^2 \\ &= \frac{m_0[u^2 + c^2(1-u^2/c^2)]}{\sqrt{1-u^2/c^2}} - m_0c^2 \\ &= \frac{m_0c^2}{\sqrt{1-u^2/c^2}} - m_0c^2 \end{aligned}$$

or

$$K = (\gamma - 1)m_0c^2. \quad (13.3)$$

This expression for kinetic energy bears little resemblance to its classical counterpart. However, in the limit $u \ll c$, $\gamma = 1/\sqrt{1-u^2/c^2} \approx \frac{1}{2}u^2/c^2$.

Using the expansion $1/\sqrt{1-x} = 1 + \frac{1}{2}x + \dots$ we obtain

$$\begin{aligned} K &\approx m_0 c^2 \left(1 + \frac{1}{2} \frac{u^2}{c^2} - 1 \right) \\ &= \frac{1}{2} m_0 u^2. \end{aligned}$$

The kinetic energy arises from the work done on the particle to bring it from rest to speed u . Using the relation $mc^2 = \gamma m_0 c^2$, we can rearrange Eq. (13.3) to give

$$\begin{aligned} mc^2 &= K + m_0 c^2 \\ &= \text{work done on particle} + m_0 c^2. \end{aligned} \quad (13.4)$$

Einstein proposed the following bold interpretation of this result: mc^2 is the *total* energy E of the particle. The first term arises from external work; the second term, $m_0 c^2$, represents the “rest” energy the particle possesses by virtue of its mass. In summary,

$$E = mc^2. \quad (13.5)$$

It is important to realize that Einstein’s generalization goes far beyond the classical conservation law for mechanical energy. Thus, if energy ΔE is added to a body, its mass will change by $\Delta m = \Delta E/c^2$, irrespective of the form of energy. ΔE could be mechanical work, heat energy, the absorption of light, or any other form of energy. In relativity the classical distinction between mechanical energy and other forms of energy disappears. Relativity treats all forms of energy on an equal footing, in contrast to Newtonian physics where each form of energy must be treated as a special case.

The conservation of total energy $E = mc^2$ is a consequence of the structure of relativity. In Chapter 14 we shall show that the conservation laws for energy and momentum are actually different aspects of a single, more general, conservation law.

The following example illustrates the relativistic concept of energy and the application of the conservation laws in different inertial frames.

Example 13.2 Relativistic Energy and Momentum in an Inelastic Collision

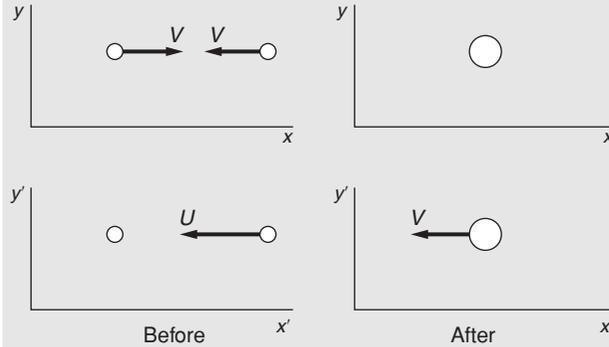
Suppose that two identical particles each of mass M collide with equal and opposite velocities and stick together. In Newtonian physics, the initial kinetic energy is $2(\frac{1}{2}MV^2) = MV^2$. By conservation of momentum the mass $2M$ is at rest and has zero kinetic energy. In the language of Chapter 4 we say that mechanical energy MV^2 was lost as heat. As we shall see, this distinction between these different classical forms of energy does not occur in relativity.

Now consider the same collision relativistically, as seen in the original frame x, y , and in the frame x', y' moving with one of the particles. By

the relativistic transformation of velocities, Eqs. (12.8) and (12.9), the relative velocity in the x' , y' frame is

$$U = \frac{2V}{1 + V^2/c^2} \quad (1)$$

in the direction shown.



Let the rest mass of each particle be M_{0i} before the collision and M_{0f} after the collision. In the x , y frame, momentum is obviously conserved. The total energy before the collision is $2M_{0i}c^2/\sqrt{1 - V^2/c^2}$, and after the collision the energy is $2M_{0f}c^2$. No external work was done on the particles, and the total energy is unchanged. Therefore

$$\frac{2M_{0i}c^2}{\sqrt{1 - V^2/c^2}} = 2M_{0f}c^2$$

or

$$M_{0f} = \frac{M_{0i}}{\sqrt{1 - V^2/c^2}}. \quad (2)$$

Physically, the final rest mass is greater than the initial rest mass because the particles are warmer. To see this, we take the low-velocity approximation

$$M_{0f} \approx M_{0i} \left(1 + \frac{1}{2} \frac{V^2}{c^2} \right).$$

The increase in rest energy for the two particles is $2(M_{0f} - M_{0i})c^2 \approx 2(\frac{1}{2}M_{0i}V^2)$, which corresponds to the loss of Newtonian kinetic energy. Now, however, the kinetic energy is not “lost”—it is present as a mass increase.

By the postulate that all inertial frames are equivalent, the conservation laws must hold in the x' , y' frame as well. Checking to see if our assumed conservation laws possess this necessary property, we have in the x' , y' frame

$$\frac{M_{0i}U}{\sqrt{1 - U^2/c^2}} = \frac{2M_{0f}V}{\sqrt{1 - V^2/c^2}} \quad (3)$$

by conservation of momentum and

$$M_{0i}c^2 + \frac{M_{0i}c^2}{\sqrt{1 - U^2/c^2}} = \frac{2M_{0f}c^2}{\sqrt{1 - V^2/c^2}} \quad (4)$$

by conservation of energy.

The question now is whether Eqs. (3) and (4) are consistent with our earlier results, Eqs. (1) and (2). To check Eq. (3), we use Eq. (1) to write

$$\begin{aligned} 1 - \frac{U^2}{c^2} &= 1 - \frac{4V^2/c^2}{(1 + V^2/c^2)^2} \\ &= \frac{(1 - V^2/c^2)^2}{(1 + V^2/c^2)^2}. \end{aligned} \quad (5)$$

From Eqs. (1) and (5),

$$\begin{aligned} \frac{U}{\sqrt{1 - U^2/c^2}} &= \frac{2V}{(1 + V^2/c^2)} \frac{(1 + V^2/c^2)}{(1 - V^2/c^2)} \\ &= \frac{2V}{1 - V^2/c^2} \end{aligned}$$

and the left-hand side of Eq. (3) becomes

$$\frac{M_{0i}U}{\sqrt{1 - U^2/c^2}} = \frac{2M_{0i}V}{1 - V^2/c^2}. \quad (6)$$

From Eq. (2), $M_{0i} = M_{0f} \sqrt{1 - V^2/c^2}$, and Eq. (6) reduces to

$$\frac{M_{0i}U}{\sqrt{1 - U^2/c^2}} = \frac{2M_{0f}V}{\sqrt{1 - V^2/c^2}},$$

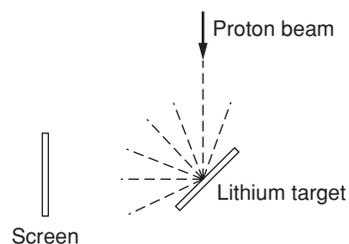
which is identical to Eq. (3). Similarly, it is not hard to show that Eq. (4) is also consistent.

We see from Eq. (6) that if we had assumed that rest mass was unchanged in the collision, $M_{0i} = M_{0f}$, the conservation law for momentum (or for energy) would not be correct in the second inertial frame. The relativistic description of energy is essential for maintaining the validity of the conservation laws in all inertial frames.

Example 13.3 The Equivalence of Mass and Energy

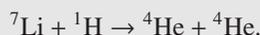
In 1932 J.D. Cockcroft and E.T.S. Walton, two young British physicists, successfully operated the first high energy proton accelerator and succeeded in causing a nuclear disintegration. Their experiment provided one of the earliest confirmations of the relativistic mass–energy relation.

Briefly, their accelerator consisted of a power supply that could reach 600 kV and a source of protons (hydrogen nuclei). The power supply used an ingenious arrangement of capacitors and rectifiers to



quadruple the voltage of a 150 kV supply. The protons were supplied by an electrical discharge in hydrogen and were accelerated in vacuum by the applied high voltage.

Cockcroft and Walton studied the effect of the protons on a target of ${}^7\text{Li}$ (lithium atomic mass 7). A zinc sulfide fluorescent screen, located nearby, emitted occasional flashes, or scintillations. By various tests they determined that the scintillations were due to alpha particles, the nuclei of helium, ${}^4\text{He}$. Their interpretation was that ${}^7\text{Li}$ captures a proton and that the resulting nucleus of mass 8 immediately disintegrates into two alpha particles. We can write the reaction as



The mass–energy equation for the reaction can be written

$$K_{\text{initial}} + M_{\text{initial}}c^2 = K_{\text{final}} + M_{\text{final}}c^2$$

where the masses are the particle rest masses. Applied to the lithium bombardment experiment, this gives

$$K({}^1\text{H}) + [M({}^1\text{H}) + M({}^7\text{Li})]c^2 = 2K({}^4\text{He}) + 2M({}^4\text{He})c^2$$

where $K({}^1\text{H})$ is the kinetic energy of the incident proton, $K({}^4\text{He})$ is the kinetic energy of each emitted alpha particle, $M({}^1\text{H})$ is the proton rest mass, etc. (The initial momentum of the proton is negligible, and the two alpha particles are emitted back-to-back with equal energy by conservation of momentum.)

We can rewrite the mass–energy equation as

$$K = \Delta Mc^2,$$

where $K = 2K({}^4\text{He}) - K({}^1\text{H})$, and where ΔM is the initial rest mass minus the final rest mass.

The energy of the alpha particles was determined by measuring their range in matter. Cockcroft and Walton obtained the value $K = 17.2 \text{ MeV}$ ($1 \text{ MeV} = 10^6 \text{ eV} = 1.6 \times 10^{-13} \text{ J}$).

The relative masses of the nuclei were known from mass spectrometer measurements. In atomic mass units, amu, the values available to Cockcroft and Walton were

$$M({}^1\text{H}) = 1.0072$$

$$M({}^7\text{Li}) = 7.0104 \pm 0.0030$$

$$M({}^4\text{He}) = 4.0011.$$

Using these values,

$$\begin{aligned} \Delta M &= (1.0072 + 7.0104) - 2(4.0011) \\ &= (0.0154 \pm 0.0030) \text{ amu.} \end{aligned}$$

The rest energy of 1 amu is ≈ 931 MeV and therefore

$$\Delta Mc^2 = (14.3 \pm 2.7)\text{MeV}.$$

The difference between K and ΔMc^2 is $(17.2 - 14.3)$ MeV = 2.9 MeV, slightly larger than the experimental uncertainty of 2.7 MeV. However, the experimental uncertainty always represents an estimate, not a precise limit, and the result from these early experiments can be taken as consistent with the relation $K = \Delta Mc^2$. It is clear that the masses must be known to high accuracy for studying the energy balance in nuclear reactions. Modern techniques of mass spectrometry have achieved an accuracy of better than 10^{-10} amu, and the mass–energy equivalence has been amply confirmed to within experimental accuracy. According to a modern table of masses, the decrease in rest mass in the reaction studied by Cockcroft and Walton is $\Delta Mc^2 = (17.3468 \pm 0.0012)$ MeV.

13.4 How Relativistic Energy and Momentum are Related

Often it is useful to express the total energy of a free particle in terms of its momentum. In Newtonian physics the relation is

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m}.$$

To find the equivalent relativistic expression we can combine the relativistic momentum

$$\mathbf{p} = m\mathbf{u} = \frac{m_0\mathbf{u}}{\sqrt{1 - u^2/c^2}} = \gamma m_0\mathbf{u} \quad (13.6)$$

with the energy

$$E = mc^2 = \gamma m_0 c^2. \quad (13.7)$$

Squaring Eq. (13.6) gives

$$p^2 = \frac{m_0^2 u^2}{1 - u^2/c^2}.$$

We can solve for γ as follows:

$$\begin{aligned} \frac{u^2}{c^2} &= \frac{p^2}{p^2 + m_0^2 c^2} \\ \gamma &= \frac{1}{\sqrt{1 - u^2/c^2}} \\ &= \sqrt{1 + \frac{p^2}{m_0^2 c^2}}. \end{aligned}$$

Inserting this in Eq. (13.7), we have

$$E = m_0 c^2 \sqrt{1 + \frac{p^2}{m_0^2 c^2}}.$$

The square of this equation is algebraically simpler and is the form usually employed:

$$E^2 = (pc)^2 + (m_0 c^2)^2. \quad (13.8)$$

For convenience, here is a summary of the important dynamical formulas we have developed so far.

$$\mathbf{p} = m\mathbf{u} = m_0 \mathbf{u} \gamma \quad (13.9)$$

$$K = mc^2 - m_0 c^2 = m_0 c^2 (\gamma - 1) \quad (13.10)$$

$$E = mc^2 = m_0 c^2 \gamma \quad (13.11)$$

$$E^2 = (pc)^2 + (m_0 c^2)^2. \quad (13.12)$$

13.5 The Photon: A Massless Particle

In 1905, in the *annus mirabilis* when Albert Einstein published four papers each worthy of a Nobel Prize, the first paper, and the only one to actually receive the Prize, had the unlikely title *On a Heuristic Viewpoint Concerning the Production and Transformation of Light*. A heuristic theory is a theory based partly on guesswork, intended to stimulate thinking. The paper ostensibly provided an explanation for the photoelectric effect, the process by which electrons are ejected from a surface when it is irradiated with light. It is now recognized that the paper provided the foundation for the quantum theory of light, contributed significantly to the development of quantum mechanics, and made applications such as the laser possible.

Because light is inherently relativistic, Einstein's paper actually opened a chapter on relativity even before relativity had been announced. At the heart of his argument is a concept that makes little sense in Newtonian physics but perfect sense in relativistic physics: *massless particles that carry momentum*.

A little background is needed: In December, 1900, quantum physics was born when Max Planck proposed that the energy of a harmonic oscillator cannot be varied at will but only by discrete steps. If the frequency of the oscillator is ν , then the energy steps had size $h\nu$ where h is a constant, now called *Planck's constant*, $h \approx 6.6 \times 10^{-34} \text{ m}^2 \text{ kg}^2/\text{s} = 6.6 \times 10^{-34} \text{ joule} \cdot \text{second}$. Planck proposed this idea to solve the mystery of thermal radiation, often called *blackbody radiation*. The shape of the spectrum of radiation emitted by a warm body could not be accounted for by the known laws of physics, based on Newtonian mechanics and Maxwell's electromagnetic theory. Planck put forward his hypothesis more in the spirit of a mathematical conjecture than

a physical theory, but in 1905 Einstein came to a similar conclusion, though by totally different reasoning, and his theory had some startling implications.

Einstein was thoroughly aware of the wave nature of light. He knew all about Maxwell's equations and how they predicted the existence of electromagnetic waves—light waves—that can travel through empty space with speed c . The wavelength of an electromagnetic wave λ and its frequency ν are related by $\lambda\nu = c$. There was a considerable body of experimental evidence that confirmed the wave nature of light, for instance the colors in soap bubble films that are a signature of light waves interfering, not to mention the fringes in Michelson's interferometer. Einstein, however, pointed out that these phenomena involve observations at the macroscopic (large scale) level. Macroscopic behavior results from the effect of many microscopic events. He pointed out that little was known about how light interacted with matter at the atomic or individual particle level. He went on to argue that light could also be understood from a particle point of view. He suggested that a light wave could behave as if it were a gas of particles, each possessing energy $\epsilon = h\nu = hc/\lambda$, where ν is the frequency of the wave. This particle hypothesis seemed to be in direct contradiction to the wave theory of light.

We now understand that light displays either wave-like or particle-like behavior depending on the situation. To understand light from the wave point of view, one starts by writing Maxwell's wave equations. Their solution reveals that time-varying electric and magnetic fields in space support each other to create an electromagnetic wave that travels at the speed of light. Furthermore, no matter which inertial coordinate system one chooses for describing the radiation process, the wave always propagates at speed c . In other words, Maxwell's equations are intrinsically relativistic. Einstein showed that we can also understand the relativistic behavior of light starting from a particle point of view, and that is the approach we now follow.

A startling consequence of the relativistic energy–momentum relation is the possibility of “massless” particles, particles that possess momentum and energy but have zero rest mass. The essential point is that a particle can possess momentum without possessing mass. This follows from the definition of relativistic momentum

$$\mathbf{p} = m_0 \mathbf{u} \left(\frac{1}{\sqrt{1 - u^2/c^2}} \right).$$

If we consider the limit $m_0 \rightarrow 0$ while $u \rightarrow c$, then \mathbf{p} can remain finite. Evidently a particle without mass can carry momentum, provided that it travels at the speed of light. From Eq. (13.12),

$$E^2 = (pc)^2 + (m_0c^2)^2,$$

and if we take $m_0 = 0$, then we have, denoting photon energy by the symbol ϵ ,

$$\begin{aligned}\epsilon^2 &= (pc)^2, \\ \epsilon &= pc.\end{aligned}\tag{13.13}$$

We have taken the positive square root because the negative solution would predict that in an isolated system the momentum of a photon could increase without limit as its energy dropped. Combining Eq. (13.13) with Einstein's relation $\epsilon = h\nu$, we find that a photon possesses momentum p of magnitude

$$p = \frac{h\nu}{c}.\tag{13.14}$$

The direction of the momentum vector is along the direction of travel of the light wave.

Einstein's quantum hypothesis was designed to solve a theoretical dilemma—the spectrum of blackbody radiation—but its first application was to a totally different problem—the photoelectric effect.

Example 13.4 The Photoelectric Effect

In 1887 Heinrich Hertz discovered that metals can give off electrons when illuminated by ultraviolet light. This process, the *photoelectric effect*, represents the direct conversion of light into mechanical energy (here, the kinetic energy of the electron). Einstein predicted that the energy a single electron absorbs from a beam of light at frequency ν is exactly the energy of a single photon, $h\nu$. For the electron to escape from the surface it must overcome the energy barrier that confines it to the surface. The electron must expend energy $W = e\Phi$ to escape from the surface, where e is the charge of the electron and Φ is an electric potential known as the *work function* of the material, typically a few volts. The maximum kinetic energy of the emitted electron is therefore

$$K = h\nu - e\Phi.$$

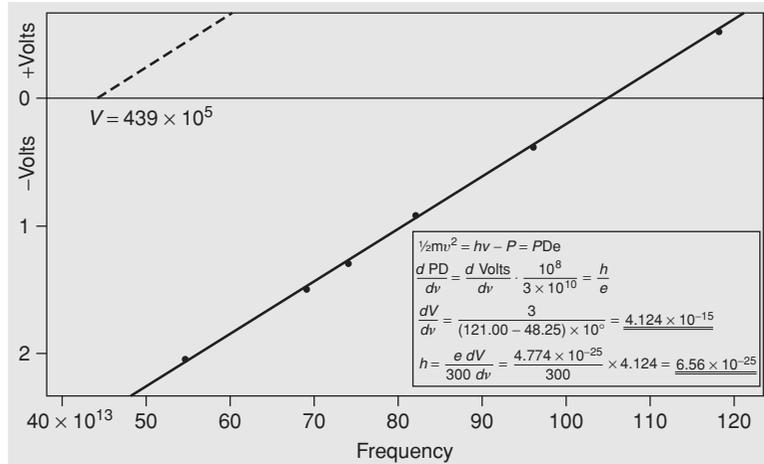
The work function depends on the poorly known chemical state of the surface, making the photoelectric effect difficult to investigate. Nevertheless, Robert A. Millikan overcame this problem in 1914 by working with metal surfaces prepared in a high vacuum system. He plotted the reverse voltage V needed to prevent the photoelectrons from reaching a detector as a function of the frequency of light. The voltage is given by

$$eV = K = h\nu - e\Phi.$$

The slope of the plot of V versus ν is

$$\frac{dV}{d\nu} = \frac{h}{e}.\tag{13.15}$$

The photoelectric effect: experimental results on the energy of photoelectrons and the frequency of light. The graph is from R.A. Millikan. From R.A. Millikan, *Physical Review* 7, 355 (1916).



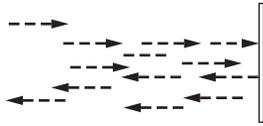
The graph of Millikan's results shows the linear relation between energy and frequency predicted by Einstein, and the slope of the line provides an accurate value for the ratio of two fundamental constants, Planck's constant and the charge of the electron.

The fact that light can interfere with itself, as in the Michelson interferometer, is compelling evidence that light has wave properties. Nevertheless, the photoelectric effect illustrates that light also has particle properties. Einstein's energy relation, $E = h\nu$, provides the link between these apparently conflicting descriptions of light by relating the energy of the photon to the frequency of the wave.

Example 13.5 The Pressure of Light

The photon picture of light provided an immediate explanation for a phenomenon that was also predicted by Maxwell's electromagnetic theory: the pressure of light. If a beam of light is absorbed or reflected by a body, it exerts a force on the body. The force per unit area, the *radiation pressure*, is too small to feel when we are in sunlight but it can have visible effects. Radiation pressure causes comets' tails to always point away from the Sun. On the astronomical scale, it helps prevent stars from collapsing under their gravitational attraction. In ultra-high intensity laser beams radiation pressure can be large enough to compress matter to the high density needed to initiate fusion reactions.

Energy flow in a light beam is often characterized by the beam's *intensity* I , which is the power per unit area of the light beam. If the number of photons crossing a unit area per second is \dot{N} and each photon carries energy ϵ , then $I = \dot{N}\epsilon$.



Consider a stream of photons in a monochromatic light beam striking a perfectly reflecting mirror at normal incidence. The initial momentum of each photon is $p = \epsilon/c$ directed toward the mirror, and the total change in momentum after the reflection is $2p = 2\epsilon/c$. The total momentum change per unit area per second due to the reflection is $2\dot{N}p = 2\dot{N}\epsilon/c$. This is the force on the light beam due to the mirror. The reaction force is the pressure P on the mirror due to the light. Hence

$$P = \frac{2\dot{N}\epsilon}{c} = \frac{2I}{c}.$$

The average intensity of sunlight falling on the Earth's surface at normal incidence, known as the *solar constant*, is $\approx 1000 \text{ W/m}^2$. The radiation pressure of sunlight on a mirror is therefore

$$\begin{aligned} P &= 2I/c \\ &= 7 \times 10^{-6} \text{ N/m}^2 \end{aligned}$$

which is very small compared, for example, to atmospheric pressure 10^5 N/m^2 .

Newtonian particles can be neither created nor destroyed. If they are combined, their total mass is constant. In contrast, massless particles can be created and annihilated. The emission of light occurs by the creation of photons, while the absorption of light occurs by the destruction of photons. The familiar laws of conservation of momentum and energy, as expressed in the theory of relativity, let us draw conclusions about processes involving photons without a detailed knowledge of the interactions, as the following examples illustrate.

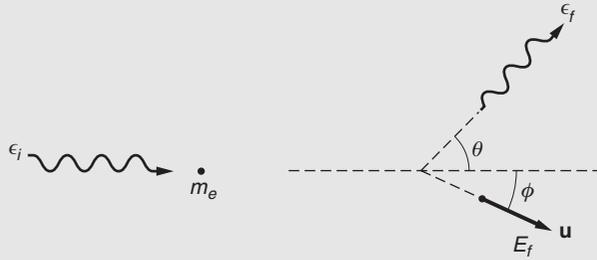
Example 13.6 The Compton Effect

The photon description of light seemed so strange that it was not widely accepted until an experiment by Arthur Compton in 1922 made the photon picture inescapable: by scattering x-rays from electrons in matter, and showing that the x-rays scattered like particles undergoing elastic collisions, and that the dynamics were correctly described by special relativity.

A photon of visible light has energy in the range of 1 to 2 eV, but photons of much higher energy can be obtained from x-ray tubes, particle accelerators, or cosmic rays. X-ray photons have energies typically in the range 10 to 100 keV. Their wavelengths can be measured with high accuracy by the technique of crystal diffraction.

When a photon scatters from a free electron, the conservation laws require that the photon loses a portion of its energy due to the recoil of the electron. The outgoing photon therefore has a longer wavelength

than the incoming photon. The shift in wavelength, first observed by Compton, is known as the *Compton effect*.



Suppose that a photon having initial energy ϵ_i and momentum ϵ_i/c is scattered at angle θ and has final energy ϵ_f . The electron has rest mass m_e and relativistic mass $m = \gamma m_e$. The electron is assumed to be initially at rest with energy $E_i = m_e c^2$. The scattered electron leaves at angle ϕ with momentum \mathbf{p} and energy $E_f = mc^2$. Here $m = m_e \gamma = m_0 / \sqrt{1 - u^2/c^2}$, where u is the speed of the recoiling electron.

The initial photon energy ϵ_i is known and the final photon energy ϵ_f and the scattering angle θ are measured. The problem is to calculate how ϵ_f varies with θ .

Conservation of total energy requires

$$\epsilon_i + m_e c^2 = \epsilon_f + E_f \quad (1)$$

and conservation of momentum requires

$$\frac{\epsilon_i}{c} = \frac{\epsilon_f}{c} \cos \theta + p \cos \phi \quad (2)$$

$$0 = \frac{\epsilon_f}{c} \sin \theta - p \sin \phi. \quad (3)$$

Because Compton detected only the outgoing photon our object is to eliminate reference to the electron and find ϵ_f as a function of θ . Equations (2) and (3) can be written

$$\begin{aligned} (\epsilon_i - \epsilon_f \cos \theta)^2 &= (pc)^2 \cos^2 \phi \\ (\epsilon_f \sin \theta)^2 &= (pc)^2 \sin^2 \phi. \end{aligned}$$

Adding,

$$\epsilon_i^2 - 2\epsilon_i \epsilon_f \cos \theta + \epsilon_f^2 = (pc)^2. \quad (4)$$

To solve for ϵ_f , we introduce the energy–momentum relation in Eq. (13.12), which can be written $(pc)^2 = (mc^2)^2 - (m_e c^2)^2$. Combining this with Eq. (4) gives

$$\epsilon_f^2 - 2\epsilon_i \epsilon_f \cos \theta + \epsilon_f^2 = (\epsilon_i + m_e c^2 - \epsilon_f)^2 - (m_e c^2)^2,$$

which reduces to

$$\epsilon_f = \frac{\epsilon_i}{1 + (\epsilon_i/m_e c^2)(1 - \cos \theta)}. \quad (5)$$

Note that the photon's final energy ϵ_f is always greater than zero, which means that a free electron cannot absorb a photon, but can scatter it.

Compton measured wavelengths rather than energies in his experiment. From the Einstein frequency condition, $\epsilon_i = h\nu_i = hc/\lambda_i$ and $\epsilon_f = hc/\lambda_f$, where λ_i and λ_f are the wavelengths of the incoming and outgoing photons, respectively. In terms of wavelength, Eq. (5) takes the simple form

$$\lambda_f = \lambda_i + \frac{h}{m_e c}(1 - \cos \theta).$$

The quantity $h/m_e c$ is known as the *Compton wavelength* λ_C of the electron and has the value

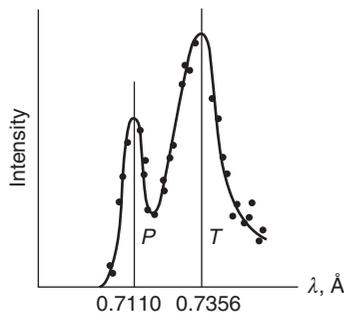
$$\begin{aligned} \lambda_C &= \frac{h}{m_e c} \\ &= 2.426 \times 10^{-12} \text{ m} \\ &= 0.02426 \text{ \AA}, \end{aligned}$$

where $1 \text{ \AA} = 10^{-10} \text{ m}$. (\AA , called the angstrom, is a non-SI unit formerly used for wavelength measurements.)

The shift in wavelength at a given angle is independent of the initial photon energy:

$$\lambda - \lambda_0 = \lambda_C(1 - \cos \theta).$$

The figure shows one of Compton's results for $\lambda_0 = 0.711 \text{ \AA}$ and $\theta = 90^\circ$. The peak *P* is due to primary photons while the peak *T* is for photons scattered from a block of graphite. The measured wavelength shift is approximately 0.0246 \AA and the calculated value is 0.02426 \AA . The difference is less than the estimated uncertainty due to the experimental limitations.



We have assumed that the electron was free and at rest. For sufficiently high photon energies, this is a good approximation for electrons in the outer shells of light atoms. If the motion of the electrons is taken into account, the Compton peak is broadened or can have structure.

If the binding energy of the electron is comparable to the photon energy, momentum and energy can be transferred to the atom as a whole, and the photon can be completely absorbed.

Example 13.7 Pair Production

We have seen two ways by which a photon can lose energy in matter: photoelectric absorption and Compton scattering. If a photon's energy is sufficiently high, it can also lose energy in matter by the mechanism of *pair production*. The rest mass of an electron is $m_0c^2 = 0.511$ MeV. Can a photon of this energy create an electron? The answer is no, since this would require the creation of a single electric charge. As far as we know, electric charge is conserved in all physical processes. However, if equal amounts of positive and negative charge are created, the total charge remains zero and charge is conserved. It is therefore possible to create an electron–positron pair (e^- , e^+), two particles having the same mass but opposite charge.

A single photon of energy $2m_0c^2$ or greater has enough energy to form an e^- , e^+ pair, but the process cannot occur in free space because it would not conserve momentum. To show why, imagine that the process occurs. Conservation of energy gives

$$h\nu = m_+c^2 + m_-c^2 = (\gamma_+ + \gamma_-)m_0c^2,$$

or

$$\frac{h\nu}{c} = (\gamma_+ + \gamma_-)m_0c,$$

while conservation of momentum gives

$$\frac{h\nu}{c} = |\gamma_+\mathbf{v}_+ + \gamma_-\mathbf{v}_-|m_0.$$

These equations cannot be satisfied simultaneously because

$$(\gamma_+ + \gamma_-)c > |\gamma_+\mathbf{v}_+ + \gamma_-\mathbf{v}_-|.$$

Pair production is possible if a third particle is available for carrying off the excess momentum. For instance, suppose that the photon collides with a nucleus of rest mass M_0 and creates an e^- , e^+ pair at rest. We have

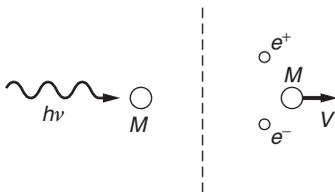
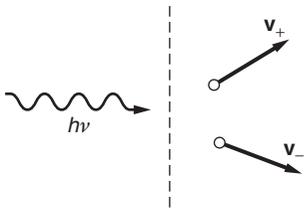
$$h\nu + M_0c^2 = 2m_0c^2 + M_0c^2\gamma.$$

Since nuclei are much more massive than electrons, let us assume that $h\nu \ll M_0c^2$. (For hydrogen, the lightest atom, this means that $h\nu \ll 940$ MeV.) In this case the atom will not attain relativistic speeds and we can make the classical approximation

$$\begin{aligned} h\nu &= 2m_0c^2 + M_0c^2(\gamma - 1) \\ &\approx 2m_0c^2 + \frac{1}{2}MV^2. \end{aligned}$$

To the same approximation, conservation of momentum yields

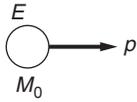
$$\frac{h\nu}{c} = MV.$$



Substituting this in the energy expression gives

$$hv = 2m_0c^2 + \frac{1}{2} \frac{(hv)^2}{Mc^2} \approx 2m_0c^2,$$

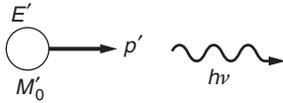
since we have already assumed $hv \ll Mc^2$. The threshold for pair production in matter is therefore $2m_0c^2 = 1.02 \text{ MeV}$. The nucleus plays an essentially passive role, but by providing for momentum conservation it allows a process to occur that would otherwise be forbidden by the conservation laws.



Example 13.8 The Photon Picture of the Doppler Effect

In Chapter 12 we analyzed the relativistic Doppler effect from the standpoint of waves but we can also understand it from the photon picture. Consider first an atom with rest mass M_0 , held stationary. If the atom emits a photon of energy $h\nu_0$, the atom's new mass is given by $M'_0c^2 = M_0c^2 - h\nu_0$.

Next, we suppose that before emitting the photon the atom moves freely with velocity \mathbf{u} . The atom's energy is $E = Mc^2 = \gamma M_0c^2$, where $\gamma = \sqrt{1 - u^2/c^2}$ and the atom's momentum is $p = Mu = M_0\gamma u$. After emitting a photon of energy $h\nu$ the atom has velocity \mathbf{u}' , rest mass M'_0 , energy E' , and momentum p' . For simplicity, we consider the photon to be emitted along the line of motion.



By conservation of energy and momentum we have

$$E = E' + h\nu \quad (1)$$

$$p = p' + \frac{h\nu}{c}. \quad (2)$$

Rearranging Eqs. (1) and (2) gives

$$(E - h\nu)^2 = E'^2$$

$$(pc - h\nu)^2 = (p'c)^2.$$

Subtracting, and using Eq. (13.12), $E^2 - (pc)^2 = (m_0c^2)^2$, we have

$$(E - h\nu)^2 - (pc - h\nu)^2 = E'^2 - (p'c)^2 = (M'_0c^2)^2 \quad (3)$$

by the energy–momentum relation. Expanding the left-hand side and using $E^2 - (pc)^2 = (M_0c^2)^2$, with $M'_0c^2 = M_0c^2 - h\nu_0$, we obtain

$$\begin{aligned} (M_0c^2)^2 - 2Eh\nu + 2(pc)(h\nu) &= (M'_0c^2)^2 \\ &= (M_0c^2 - h\nu_0)^2. \end{aligned}$$

Simplifying, we find

$$v = v_0 \frac{(2M_0c^2 - h\nu_0)}{2(E - pc)}.$$

However,

$$\begin{aligned} E - pc &= M_0 c^2 \gamma \left(1 - \frac{u}{c}\right) \\ &= M_0 c^2 \sqrt{\frac{1 - u/c}{1 + u/c}}. \end{aligned}$$

Hence

$$\nu = \nu_0 \left(1 - \frac{h\nu_0}{2M_0 c^2}\right) \sqrt{\frac{1 + u/c}{1 - u/c}}.$$

The term $h\nu_0/2M_0 c^2$ represents a decrease in the photon energy due to the recoil energy of the atom. Usually the recoil energy is so small that it can be neglected, leaving

$$\nu = \nu_0 \sqrt{\frac{1 + u/c}{1 - u/c}},$$

in agreement with the wave analysis that led to Eq. (12.12). However, the wave picture does not readily take into account the recoil of the atom. In modern experiments using high precision lasers and ultra-cold atoms, the recoil cannot be overlooked. On the contrary, it plays a crucial role in many studies.

Example 13.9 The Photon Picture of the Gravitational Red Shift

In Chapter 9 we derived an expression for the effect of gravity on time—the gravitational red shift—by invoking the equivalence principle. However, the effect of gravity on time can also be understood using the photon description of light and the conservation of energy.

Atoms can absorb or emit photons at certain characteristic frequencies. For a frequency ν_0 , the atom loses energy $h\nu_0$ when it emits a photon, going from an upper energy state E_1 to a lower energy state, E_0 , and it can gain energy $h\nu_0$ when it absorbs a photon, reversing the process.

Consider an atom with rest mass M_0 in its ground state with energy $E_0 = M_0 c^2$, in a gravitational field g . It absorbs a photon that increases its energy to $E_1 = E_0 + h\nu_0$. The mass of the atom is $M_1 = E_1/c^2 = (E_0 + h\nu_0)/c^2$. If we lift the atom to height H in a gravitational field g the work that we do is $M_1 g H$, so the final energy W_a of the atom is

$$\begin{aligned} W_a &= E_1 + M_1 g H \\ &= (E_0 + h\nu_0)(1 + gH/c^2) \\ &= E_0 + h\nu_0 + h\nu_0 gH/c^2 + E_0 gH/c^2. \end{aligned}$$

Consider an alternative scenario: the atom is first lifted to height H while it is in state E_0 , and then a photon of energy $h\nu$ is radiated upward to put the atom in state E_1 . The energy W_b of the atom with this procedure is

$$W_b = E_1 + M_0gH = E_0 + h\nu + E_0gH/c^2.$$

The final state of the system is the same in both scenarios. Consequently $W_a = W_b$ and it follows that

$$h\nu = h\nu_0(1 + gH/c^2).$$

In fractional form, the gravitational red shift is

$$\frac{\nu - \nu_0}{\nu_0} = \frac{gH}{c^2}.$$

A word of explanation about the adjective “red.” Our result reveals that if radiation travels outward from the Earth to a region of higher (less negative) gravitational potential, its energy decreases. Consequently, radiation emitted by a massive body such as the Sun is observed to shift to lower energy, equivalently to longer wavelengths, toward the red end of the spectrum. In contrast, radiation that comes down to the Earth from a satellite, for instance the signal from an atomic clock, is shifted to higher energy, which might be called a blue shift.

13.6 How Einstein Derived $E = mc^2$

Einstein’s famous equation $E = mc^2$ is not to be found in his historic paper on relativity but only appeared a few months later in a short note titled *Does the Inertia of a Body Depend Upon Its Energy Content?* (translated from German). His argument was elegant in its simplicity, based entirely on elementary considerations of energy, momentum, and the Doppler shift.

Consider a body in system S at rest at the origin. The body has energy E_0 initially, and then sends out a pulse of light with energy $\epsilon/2$ in the $+x$ direction, and simultaneously a pulse with energy $\epsilon/2$ in the $-x$ direction. The body remains at rest after the emission by conservation of momentum, and its energy is then E_1

$$E_0 = E_1 + \frac{1}{2}\epsilon + \frac{1}{2}\epsilon.$$

In system S' moving with velocity v with respect to S , the initial energy of the body is H_0 and its energy after the emission is H_1 . Taking the Doppler shift into account,

$$\begin{aligned} H_0 &= H_1 + \frac{1}{2}\epsilon \left(\frac{1 - v/c}{\sqrt{1 - v^2/c^2}} \right) + \frac{1}{2}\epsilon \left(\frac{1 + v/c}{\sqrt{1 - v^2/c^2}} \right) \\ &= H_1 + \frac{\epsilon}{\sqrt{1 - v^2/c^2}}. \end{aligned}$$

The energy differences in the two systems are

$$(H_0 - E_0) - (H_1 - E_1) = \epsilon \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right).$$

Einstein argued that the difference $H - E$ must equal the kinetic energy K of the body, to within an additive constant C that is independent of the relative velocity

$$H_0 - E_0 = K_0 + C$$

$$H_1 - E_1 = K_1 + C.$$

Thus

$$\begin{aligned} K_0 - K_1 &= \epsilon \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right) \\ &\approx \frac{1}{2} \epsilon \frac{v^2}{c^2}. \end{aligned}$$

Classically,

$$K_0 - K_1 = \frac{1}{2} \Delta m v^2.$$

Einstein then obtained his famous equation by comparing the two results for $K_0 - K_1$:

$$\Delta m = \frac{\epsilon}{c^2}.$$

Einstein concluded his brief paper by asserting that the equivalence of mass and energy must be a general law, holding for any form of energy, not just radiation.

Problems

For problems marked *, refer to page 525 for a hint, clue, or answer.

13.1 Energetic proton

Cosmic ray primary protons with energy up to 10^{20} eV (almost 10 J) have been detected. Our galaxy has a diameter of about 10^5 light years.

(a) How long does it take the proton to traverse the galaxy, in its own rest frame (proper time)? (1 eV = 1.6×10^{-19} J, $M_p = 1.67 \times 10^{-27}$ kg.) What is the proper time for a photon to traverse our galaxy?

(b) Compare the proton's energy to the kinetic energy of a baseball, mass = 145 g, traveling at 100 miles/hour.

13.2 Onset of relativistic effects

When working with particles it is important to know when relativistic effects have to be considered.

A particle of rest mass m_0 is moving with speed v . Its classical kinetic energy is $K_{\text{cl}} = m_0 v^2/2$. Let K_{rel} be the relativistic expression for its kinetic energy.

(a) By expanding $K_{\text{rel}}/K_{\text{cl}}$ in powers of v^2/c^2 , estimate the value of v^2/c^2 for which K_{rel} differs from K_{cl} by 10 percent.

(b) For this value of v^2/c^2 , what is the kinetic energy in MeV of

(1) an electron ($m_0 c^2 = 0.51$ MeV)?

(2) a proton ($m_0 c^2 = 930$ MeV)?

13.3 Momentum and energy

In Newtonian mechanics, the kinetic energy of a mass m moving with velocity \mathbf{v} is $K = mv^2/2 = p^2/(2m)$ where $\mathbf{p} = m\mathbf{v}$. The change in kinetic energy due to a small change in momentum is $dK = \mathbf{p} \cdot d\mathbf{p}/m = \mathbf{v} \cdot d\mathbf{p}$.

Show that the relation $dK = \mathbf{v} \cdot d\mathbf{p}$ also holds in relativistic mechanics.

13.4 Particles approaching head-on*

Two particles of rest mass m_0 approach each other with equal and opposite velocity v in the laboratory frame. What is the total energy of one particle as measured in the rest frame of the other?

13.5 Speed of a composite particle after an inelastic collision*

A particle of rest mass m_0 and speed v collides with a stationary particle of mass M and sticks to it. What is the final speed of the composite particle?

13.6 Rest mass of a composite particle*

A particle of rest mass m_0 and kinetic energy xm_0c^2 , where x is some number, strikes an identical particle at rest and sticks to it. What is the rest mass of the resultant particle?

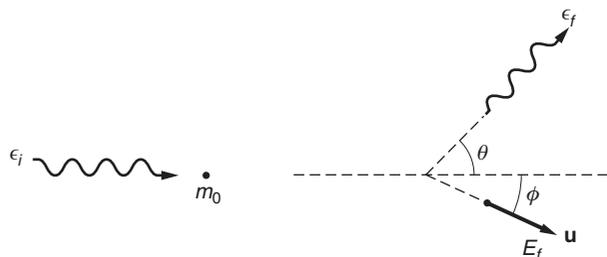
13.7 Zero momentum frame*

In the laboratory frame a particle of rest mass m_0 and speed v is moving toward a particle of mass m_0 at rest.

What is the speed of the inertial frame in which the total momentum of the system is zero?

13.8 Photon-particle scattering*

A photon of energy ϵ_i collides with a free particle of mass m_0 at rest. If the scattered photon flies off at angle θ , what is the scattering angle ϕ of the particle?





13.9 *Photon–electron collision**

A photon of energy E_0 and wavelength λ_0 collides head-on with a free electron of rest mass m_0 and speed V , as shown. The photon is scattered at 90° .

(a) Find the energy E of the scattered photon.

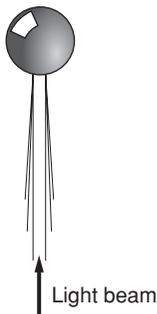
(b) The outer electrons in a carbon atom move with speed $v/c \approx 6 \times 10^{-3}$. Using the result of part (a), estimate the broadening in wavelength of the Compton scattered peak from graphite for $\lambda_0 = 0.711 \times 10^{-10}$ m and 90° scattering. The rest mass of an electron is 0.51 MeV and $h/(m_0c) = 2.426 \times 10^{-12}$ m. Neglect the binding of the electrons. Compare your result with Compton’s data shown in Example 13.6.

13.10 *The force of sunlight*

The solar constant, the average energy per unit area from the Sun falling on the Earth, is 1.4×10^3 W/m².

(a) How does the total force of sunlight compare with the Sun’s gravitational force on the Earth?

(b) Sufficiently small particles can be ejected from the solar system by the radiation pressure of sunlight. Assuming a specific gravity of 5, what is the radius of the largest particle that can be ejected?



13.11 *Levitation by laser light*

A 1-kW light beam from a laser is used to levitate a solid aluminum sphere by focusing it on the sphere from below. What is the diameter of the sphere, assuming that it floats freely in the light beam? The density of aluminum is 2.7 g/cm³.

13.12 *Final velocity of a scattered particle*

A photon of energy $\epsilon_i = h\nu$ scatters from a free particle at rest of mass m_0 . The photon is scattered at angle θ with energy $\epsilon_f = h\nu'$, and the particle flies off at angle ϕ .

Find an expression for the final velocity \mathbf{u} of the particle.

14

SPACETIME PHYSICS

- 14.1 Introduction 504
- 14.2 Vector Transformations 504
 - 14.2.1 Invariants and Scalars 505
- 14.3 World Lines in Spacetime 506
- 14.4 An Invariant in Spacetime 508
 - 14.4.1 Spacelike and Timelike Intervals 508
- 14.5 Four-Vectors 509
 - 14.5.1 Lorentz Invariants 510
 - 14.5.2 Four-Velocity 510
- 14.6 The Energy–Momentum Four-Vector 512
- 14.7 Epilogue: General Relativity 513
- Problems* 515

14.1 Introduction

In 1908, three years after Einstein published the special theory of relativity, the mathematician Hermann Minkowski presented a geometrical formulation of Einstein's ideas based on the concept of a four-dimensional manifold that he called "spacetime." Minkowski famously asserted "Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." His claim may be a little exaggerated—we continue to move freely in a three-dimensional world while being swept forward relentlessly in time—but his point of view has been invaluable in extending the concepts of relativity to other areas of physics.

Special relativity provides an orderly procedure for relating the coordinates of events recorded by observers in different inertial systems. The essence of the theory is embodied by the Lorentz transformation. To set the stage for Minkowski's spacetime description of this transformation, let's briefly review how vectors transform in Newtonian physics.

14.2 Vector Transformations

We are interested here in the transformation properties of vectors, for instance some vector \mathbf{A} that could represent a physical quantity such as force or velocity, or simply be an abstract mathematical quantity. To describe \mathbf{A} in component form we introduce an orthogonal coordinate system S with coordinates (x, y, z) and unit base vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$. \mathbf{A} can then be written

$$\mathbf{A} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}. \quad (14.1)$$

The coordinate system is not fundamental but merely a construct that we introduce for convenience. We could use some other orthogonal coordinate system S' with coordinates (x', y', z') and base vectors $(\hat{\mathbf{i}}', \hat{\mathbf{j}}', \hat{\mathbf{k}}')$. If the two systems have the same origin, they must be related by a rotation. In the primed system,

$$\mathbf{A} = A'_x \hat{\mathbf{i}}' + A'_y \hat{\mathbf{j}}' + A'_z \hat{\mathbf{k}}'. \quad (14.2)$$

Because Eqs. (14.1) and (14.2) describe the same vector, we have

$$A'_x \hat{\mathbf{i}}' + A'_y \hat{\mathbf{j}}' + A'_z \hat{\mathbf{k}}' = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}. \quad (14.3)$$

To find the coordinates in S' given the coordinates in S , take the dot product of both sides of Eq. (14.3) with the corresponding unit vector:

$$A'_x = \mathbf{A} \cdot \hat{\mathbf{i}}' = A_x (\hat{\mathbf{i}} \cdot \hat{\mathbf{i}}') + A_y (\hat{\mathbf{j}} \cdot \hat{\mathbf{i}}') + A_z (\hat{\mathbf{k}} \cdot \hat{\mathbf{i}}') \quad (14.4a)$$

$$A'_y = \mathbf{A} \cdot \hat{\mathbf{j}}' = A_x (\hat{\mathbf{i}} \cdot \hat{\mathbf{j}}') + A_y (\hat{\mathbf{j}} \cdot \hat{\mathbf{j}}') + A_z (\hat{\mathbf{k}} \cdot \hat{\mathbf{j}}') \quad (14.4b)$$

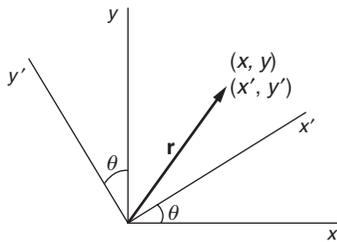
$$A'_z = \mathbf{A} \cdot \hat{\mathbf{k}}' = A_x (\hat{\mathbf{i}} \cdot \hat{\mathbf{k}}') + A_y (\hat{\mathbf{j}} \cdot \hat{\mathbf{k}}') + A_z (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'). \quad (14.4c)$$

The coefficients $(\hat{\mathbf{i}} \cdot \hat{\mathbf{i}}'), (\hat{\mathbf{j}} \cdot \hat{\mathbf{j}}')$, etc. are numbers that can be calculated for any given rotation. For instance, for rotation by θ around the z axis,

$$A'_x = A_x \cos \theta + A_y \sin \theta \quad (14.5a)$$

$$A'_y = -A_x \sin \theta + A_y \cos \theta \quad (14.5b)$$

$$A'_z = A_z. \quad (14.5c)$$



As an example, if we let \mathbf{A} be the position vector \mathbf{r} for a point that has coordinates (x, y, z) in system S and (x', y', z') in system S' rotated by angle θ around the z axis, then we have

$$x' = x \cos \theta + y \sin \theta \quad (14.6a)$$

$$y' = -x \sin \theta + y \cos \theta \quad (14.6b)$$

$$z' = z. \quad (14.6c)$$

Note that the x' and y' axes are both rotated in the same direction from the respective x and y axes. This is a trivial observation for rotations in three-dimensional space, but we will soon see that rotations in spacetime behave quite differently.

The transformation from S' back to S , by rotating the axes through angle $-\theta$, known as the *inverse transformation*, is

$$x = x' \cos \theta - y' \sin \theta \quad (14.7a)$$

$$y = x' \sin \theta + y' \cos \theta \quad (14.7b)$$

$$z = z'. \quad (14.7c)$$

Rotating axes through angle θ has the same effect on a vector's components as keeping the axes fixed and rotating the vector through angle $-\theta$. However, in this chapter we will always keep the vector fixed and rotate the axes.

14.2.1 Invariants and Scalars

Quantities that remain constant when a coordinate system changes are called *invariants*. Clearly, the components of a vector are not invariants but the vector itself is. Another invariant is the length of the vector $A = |\mathbf{A}|$ defined by

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{A_x'^2 + A_y'^2 + A_z'^2}.$$

Quantities that do not change with a change in coordinate systems are called *scalars*. Simple numbers such as mass, temperature, and Avogadro's number are scalars. The lengths of vectors are also scalars. Many physical quantities are scalars, for instance mass, time intervals, and speed (as contrasted to velocity). Such scalars play important roles in Newtonian physics and they continue to play important roles in relativity, though their interpretation is different.

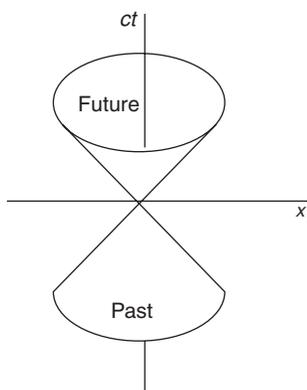
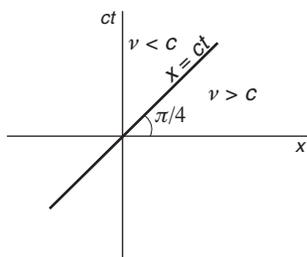
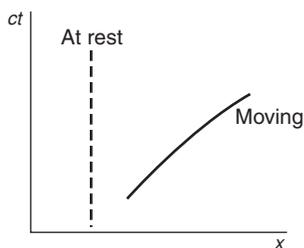
14.3 World Lines in Spacetime

To start our exploration of the spacetime world let's see how events evolve on a spacetime plot.

Motion in three-space is unrestricted. As time evolves, the point $\mathbf{r} = (x, y, z)$ can be anywhere. All of space is accessible to an observer, and all time in the future is accessible to observation, given enough time. Time, of course, moves in one direction only and is out of our control.

In spacetime physics, an event is a physical happening that can be specified by values for three spatial coordinates plus the time, (x, y, z, t) . The emission of a pulse of light from the origin at $t = 0$ is for example an event with coordinates $(0, 0, 0, 0)$. Because coordinates with different physical dimensions are awkward to use, we will write time in units of length by multiplying it by a velocity. For this we naturally multiply time by the velocity of light so that $t \rightarrow ct$. Hence $1 \mu\text{s}$ corresponds to \approx "300 m of time" and "1 m of time" corresponds to $\approx 0.0033 \mu\text{s}$. With this convention we can speak of a four-dimensional spacetime with coordinates (x, y, z, ct) .

Focusing on linear motion with coordinates x and ct , we can plot the evolution of an event on a spacetime diagram. By an awkward but universally accepted convention, time in spacetime diagrams is plotted vertically. As time evolves, a point in spacetime traces an upward path called a *world line*. The sketch shows two world lines: the vertical dashed line is the world line for a particle at rest (x constant), and the solid line is for a moving particle (x and t both increasing).



Note that with this convention, speed—in units of c —is given by the cotangent of the slope. A slope of ± 1 corresponds to a speed of $\pm c$. The fastest event is a pulse of light whose world line is given by $x = ct$ or $x = -ct$. This line is at an angle $\pi/4$ with respect to the horizontal, as shown. A world line at an angle less than $\pi/4$ would describe motion faster than light, which is prohibited. A three-dimensional plot that shows (x, y, ct) is in the form of two cones with their apexes at the origin, called *light cones*.

All future events lie in the upper light cone, past events in the lower light cone. The region of spacetime outside the light cones is physically inaccessible to an observer at the origin. Other events would be described by other light cones but for two events to be causally related their light cones must overlap.

We now turn to the question of how spacetime events appear to observers in different inertial systems. In Chapter 12 we derived the Lorentz transformation for changing coordinates between our standard inertial systems S and S' . The origin of S' moves at speed v along the x axis. Alternatively, the origin of S moves with speed $-v$ along the x' axis. Because time is now expressed in units of length ct , it is natural to express the relative velocity of the coordinate systems by the variable $\beta = v/c$. With this notation, the Lorentz transformation from S to S' , Eqs. (12.3)

and (12.4), is

$$x' = \gamma(x - \beta ct) \tag{14.8a}$$

$$y' = y \tag{14.8b}$$

$$z' = z \tag{14.8c}$$

$$ct' = \gamma(-\beta x + ct). \tag{14.8d}$$

The reverse transformation from S' to S is given by

$$x = \gamma(x' + \beta ct') \tag{14.9a}$$

$$y = y' \tag{14.9b}$$

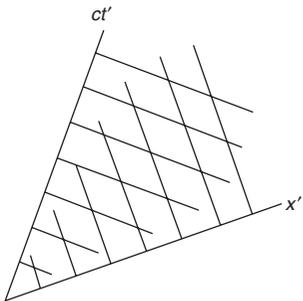
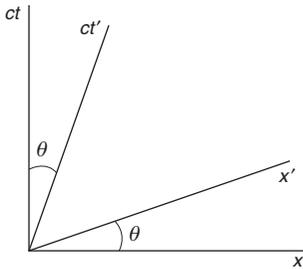
$$z = z' \tag{14.9c}$$

$$ct = \gamma(\beta x' + ct') \tag{14.9d}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1 - \beta^2}$. The quantity β is always taken to be ≥ 0 , with algebraic signs shown explicitly. Because the y and z coordinates are unchanged by the Lorentz transformation, we will concern ourselves only with x and ct .

There is a parallel between the rotation of coordinates in three-space and transformation in spacetime specified by Eqs. (14.8) and (14.9).

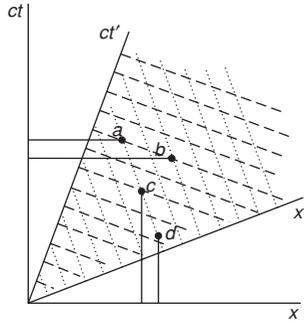
Rotations in three-space	Lorentz transformation
$x' = x \cos \theta + y \sin \theta$	$x' = \gamma(x - \beta ct)$
$y' = -x \sin \theta + y \cos \theta$	$ct' = \gamma(-\beta x + ct)$
$z' = z$	$y' = y$
$t' = t$	$z' = z$



The Lorentz transformation is similar in form to the transformation of x and y due to a rotation around the z axis. Equation (14.8d) shows that the locus of the x' axis (for which $ct' = 0$), when plotted in the x - ct plane, is given by $ct = \beta x$. This describes a *counterclockwise* rotation of the x' axis from the x axis through angle $\theta = \arctan \beta$. In contrast, Eq. (14.8a) reveals that the ct' axis (for which $x' = 0$) is given by $x = \beta ct$, which describes a *clockwise* rotation of the ct' axis from the ct axis, through the same angle. In the limiting case $v = c$, $\theta = \pi/4$; the x' and ct' axes become coincident because $x' = ct'$.

According to these transformations, the axes are no longer orthogonal. The lines of constant x' and constant ct' in the figure form a grid of diamonds rather than squares. Furthermore, the scale of the coordinate axes is changed by the factor of γ . These are fundamental differences between the geometries of three-space and spacetime.

The loss of orthogonality is a characteristic feature of transformations in spacetime. Although we can describe time as a fourth dimension, time is fundamentally different from the spatial dimensions, and that difference is crucial to the geometry of spacetime. The length-scales in a spacetime diagram differ by a factor of γ and the appearance of a world



line depends on the relative motion of the observer. Consequently, every observer would describe events with a different diagram, so that care is needed in extracting geometrical relationships from a diagram.

The drawings show some events in spacetime in the systems $S(x, ct)$ and $S'(x', ct')$. In S' , lines of constant time are dashed and lines of constant position are dotted. Notice that events a and b are coincident in S' but occur at different times in S . Similarly, events c and d occur at the same location in S' but at different locations in S .

14.4 An Invariant in Spacetime

In three-space, the length r of the position vector is $r = \sqrt{x^2 + y^2 + z^2}$, and is invariant under rotation. In spacetime, the quantity $x^2 + y^2 + z^2 - (ct)^2$ is invariant under the Lorentz transformation. To prove this, we have from Eqs. (14.8)

$$\begin{aligned} x'^2 + y'^2 + z'^2 - (ct')^2 &= \gamma^2(x - \beta ct)^2 + \gamma^2(-\beta x + ct)^2 + y^2 + z^2 \\ &= \frac{1}{1 - \beta^2} [x^2(1 - \beta^2) - (ct)^2(1 - \beta^2)] + y^2 + z^2 \\ &= x^2 + y^2 + z^2 - (ct)^2. \end{aligned}$$

Hence

$$x'^2 + y'^2 + z'^2 - (ct')^2 = x^2 + y^2 + z^2 - (ct)^2. \quad (14.10)$$

Consider a world line between an event starting at $\mathbf{R}_1 = (\mathbf{r}, ct)$ and ending at $\mathbf{R}_2 = (\mathbf{r} + \Delta\mathbf{r}, ct + \Delta ct)$. The displacement between the events is $\Delta\mathbf{R} = \mathbf{R}_2 - \mathbf{R}_1 = (\Delta\mathbf{r}, \Delta ct)$, where $\Delta\mathbf{r} = (\Delta x, \Delta y, \Delta z)$. The coordinates for these displacements when viewed by an observer in S' are $\Delta\mathbf{r}'$ and $\Delta ct'$, which can be found using the Lorentz transformation. From Eq. (14.10) we have

$$\Delta r'^2 - \Delta(ct')^2 = \Delta r^2 - \Delta(ct)^2.$$

Note that by convention in relativity, Δx^2 is interpreted as $(\Delta x)^2$, not $\Delta(x^2)$. Because the Lorentz transformation depends on the relative velocity v/c , and because the relative velocity can be chosen arbitrarily, the only way that this equation can be satisfied is for the two sides to separately equal a constant. Denoting this constant by Δs^2 , we have

$$\Delta s^2 \equiv \Delta r^2 - \Delta(ct)^2. \quad (14.11)$$

Consequently, Δs^2 is an invariant of the transformation. Δs^2 is often called the *separation* of the spacetime interval. Unlike the square of an ordinary number, $(\Delta s)^2$ can be negative.

14.4.1 Spacelike and Timelike Intervals

If $\Delta s^2 > 0$, we can always find a coordinate system that satisfies $\Delta r^2 = \Delta s^2$, with $\Delta(ct)^2 = 0$. Consequently, there is a coordinate system in which the events are simultaneous, but there is no frame in which

the events are coincident in space. Such an interval is called *spacelike*. Similarly, if $\Delta s^2 < 0$, there is a coordinate system in which the events can coincide in space, but there is no frame in which they occur simultaneously. Such an interval is called *timelike*. If $\Delta s^2 = 0$, the events correspond to the emission and reception of a pulse of light. The world line obeys $\Delta r = \Delta ct$ and the separation is called *null*. A separation has the same value in all inertial systems.

For a given spacetime interval, the geometry of possible displacements looks entirely different from the geometry of a three-space interval, where the displacements of a given magnitude lie on a sphere. In two dimensions, the locus of possible displacement is a circle, $\Delta r^2 = \Delta x^2 + \Delta y^2$. In spacetime, the locus of an interval for the (x, y, ct) coordinates is a hyperbolic surface of revolution.

$$x^2 + y^2 - (ct)^2 = \pm |\Delta s^2|. \quad (14.12)$$

14.5 Four-Vectors

We developed the Lorentz transformation to satisfy the need for a spacetime transformation that preserves the speed of light as a universal constant. The reasoning behind the Lorentz transformation is more powerful than this application might suggest. The ideas can be extended to develop relativistic dynamics in a coherent fashion by general arguments that can apply to new systems, for example to the dynamics of electric and magnetic fields, though we will not pursue that development here. The starting point for this generalization is the concept of a spacetime vector, usually referred to as a *four-vector*.

A four-vector is a set of four numbers that transform according to the Lorentz transformation. Such vectors have the potential to be physically significant. Put another way, any set of four numbers that do not obey the Lorentz transformation cannot enter into a physical law because the law could not satisfy the principle of relativity. A good way to extend our understanding of physics in the relativistic world is by searching for four-vectors. A four-vector \mathbf{A} has the general form

$$\mathbf{A} = (a_1, a_2, a_3, a_4)$$

where the components (a_1, a_2, a_3) are along the axes of the standard system $S = (x, y, z)$. Under a Lorentz transformation to a coordinate system S' moving in the x direction with velocity v , $\mathbf{A}_4 = (a'_1, a'_2, a'_3, a'_4)$, where

$$a'_1 = \gamma(a_1 - \beta a_4) \quad (14.13a)$$

$$a'_2 = a_2 \quad (14.13b)$$

$$a'_3 = a_3 \quad (14.13c)$$

$$a'_4 = \gamma(-\beta a_1 + a_4). \quad (14.13d)$$

As previously defined, $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$. We shall use the convention that four-vectors are printed in upper case bold, while

three-vectors are in lower case bold. Thus, the four-vector for four-position \mathbf{R} can be written $\mathbf{R} = (\mathbf{r}, ct) = (x, y, z, ct)$.

14.5.1 Lorentz Invariants

The sum of the squares of the components of a four-vector, with a negative sign for the square of the time component, is called the *norm* of the four-vector. For example, the norm of $\mathbf{A} = (a_1, a_2, a_3, a_4)$ is $a_1^2 + a_2^2 + a_3^2 - a_4^2$.

A true four-vector transforms according to the Lorentz transformation, as in Eqs. (14.13). It is easy to show that the norm is the same in any frame, and the norm is therefore called a *Lorentz invariant*.

$$a_1'^2 + a_2'^2 + a_3'^2 - a_4'^2 = a_1^2 + a_2^2 + a_3^2 - a_4^2.$$

We showed this invariance for the four-vector \mathbf{R} in Section 14.4.

14.5.2 Four-Velocity

The simplest kinematical quantity after position is velocity, and so it is natural to consider the rate of change of \mathbf{R} . However, the concept of *rate* requires the introduction of a clock, and the question then is whose clock? The only time on which all observers could agree is the proper time τ of a clock attached to the moving point. Of course, this would differ from the observer's time, but by a known amount.

Consequently, we can tentatively define the *four-velocity* \mathbf{U} by

$$\mathbf{U} \equiv \frac{d\mathbf{R}}{d\tau} = \left(\frac{d\mathbf{r}}{d\tau}, \frac{d(ct)}{d\tau} \right). \quad (14.14)$$

To connect the spatial part with the familiar three-velocity of a Newtonian observer, we can rewrite this as

$$\mathbf{U} = \left(\frac{d\mathbf{r}}{dt} \frac{dt}{d\tau}, \frac{d(ct)}{d\tau} \right).$$

From the discussion of proper time in Section 12.8.3, the relation between a local time interval and the proper time interval is

$$d\tau = dt \sqrt{1 - v^2/c^2} = dt/\gamma. \quad (14.15)$$

Hence $dt/d\tau = \gamma$ and $\mathbf{U} = (\gamma\mathbf{u}, \gamma c)$, or

$$\mathbf{U} = \gamma(\mathbf{u}, c). \quad (14.16)$$

Consider the norm of the four-velocity $\mathbf{U} = \gamma(\mathbf{u}, c)$:

$$\begin{aligned} \mathbf{U} \cdot \mathbf{U} &= \gamma^2(u^2 - c^2) \\ &= -c^2. \end{aligned}$$

The norm of $\mathbf{U} = -c^2$ is obviously a Lorentz invariant, the same in every reference frame.

Example 14.1 Relativistic Addition of Velocities

The relativistic law for the addition of velocities was derived in Section 12.9. Here we look at the same problem using the four-velocity approach.

Consider for simplicity velocity along the x axis. If the initial speed is u_1 along x , then we can take the initial four-velocity to be $\mathbf{U} = \gamma(u_1, c)$. If we observe an event with this velocity in system S_1 moving with speed v_1 , then the Lorentz transformation Eqs. (14.8) shows that the components of the four-velocity are

$$u'_1 = \gamma_1(u_1 - \beta_1 c) \quad (1a)$$

$$u'_4 = \gamma_1(-\beta_1 u_1 + c), \quad (1b)$$

where $\beta_1 = v_1/c$ and $\gamma_1 = 1/\sqrt{1 - \beta_1^2}$.

If we now move into system S_2 moving with speed v_2 relative to S_1 , we have

$$u''_1 = \gamma_2(u'_1 - \beta_2 u'_4) \quad (2a)$$

$$u''_4 = \gamma_2(-\beta_2 u'_1 + u'_4), \quad (2b)$$

where $\beta_2 = v_2/c$ and $\gamma_2 = 1/\sqrt{1 - \beta_2^2}$. These two successive transformations are equivalent to a single transformation found by substituting Eq. (1) in Eq. (2). The result, after some rearranging, is

$$u''_1 = \frac{\gamma_2 \gamma_1}{1 + \beta_1 \beta_2} \left(u_1 - \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} c \right) \quad (3a)$$

$$u''_4 = \frac{\gamma_2 \gamma_1}{1 + \beta_1 \beta_2} \left(-\frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} u_1 + u_4 \right). \quad (3b)$$

Comparing Eq. (3) with Eq. (1), we see that the form is equivalent to moving into a coordinate system with speed β_3 given by

$$\beta_3 = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}$$

or, in laboratory units,

$$v_3 = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}.$$

This is the rule for the relativistic addition of velocities that we found in Section 12.9.

One might reasonably ask what purpose is served by introducing the four-velocity $\mathbf{U} = \gamma(\mathbf{v}, c)$, because the timelike component is simply a constant, c . But as we shall see in the next section, the constant plays an important role.

14.6 The Energy–Momentum Four-Vector

The natural next step is to define the four-momentum \mathbf{P} of a particle of mass m moving with four-velocity \mathbf{U} . Using Eq. (14.16), we have

$$\mathbf{P} = m\mathbf{U} = \gamma m_0(\mathbf{u}, c). \quad (14.17)$$

Here m_0 is the mass observed when $u \ll c$ and $\gamma = 1/\sqrt{1 - u^2/c^2} \approx 1$. In Chapter 13 we employed the term *rest mass* for m_0 . This quantity would more logically be called the *proper mass* since it is the mass measured in the particle's rest frame but because of historical circumstance the term *proper mass* is not used. In relativity, the simple term *mass* always refers to the *relativistic mass* m defined by

$$m \equiv \gamma m_0. \quad (14.18)$$

Because \mathbf{U} is a four-vector and m is a scalar, $\mathbf{P} = m\mathbf{U}$ is also a four-vector. The four-momentum can be written

$$\mathbf{P} = (m\mathbf{U}, mc) = (\mathbf{p}, mc), \quad (14.19)$$

where $\mathbf{p} = m\mathbf{u}$ is the relativistic three-momentum. The norm of \mathbf{P} is

$$\mathbf{P} \cdot \mathbf{P} = \mathbf{p} \cdot \mathbf{p} - (mc)^2 = C \quad (14.20)$$

where C is a constant. In the particle's rest frame $\mathbf{p} = 0$, $C = -(m_0c)^2$. Thus the norm of the four-momentum is $-(m_0c)^2$ and we can rewrite Eq. (14.20) as

$$p^2 = (mc)^2 - (m_0c)^2. \quad (14.21)$$

To be useful, four-momentum must be conserved in an isolated system. Three-momentum vanishes in a system in which a particle is at rest. Consequently, in Eq. (14.20) \mathbf{p} and mc must be separately conserved. To make connections with Newtonian concepts, recall that in Newtonian physics, in addition to the linear momentum of an isolated system, energy and angular momentum are also conserved. Four-momentum has no evident connection with angular momentum so let us guess that mc , the temporal component of the four-vector in Eq. (14.19), is related to the energy E . To be dimensionally correct, we tentatively set

$$mc = E/c \quad (14.22)$$

so that \mathbf{P} takes the form

$$\mathbf{P} = (\mathbf{p}, E/c). \quad (14.23)$$

\mathbf{P} is sometimes called the *energy-momentum four-vector*, but more often it is referred to simply as the *four-momentum*. Introducing Eq. (14.22) into Eq. (14.21), we have

$$(pc)^2 = E^2 - (m_0c^2)^2 \quad (14.24)$$

or

$$E^2 = (pc)^2 + (m_0c^2)^2, \quad (14.25)$$

a result we found in Chapter 13, Eq. (13.8). The term m_0c^2 is known as the *rest energy* of the particle.

For a massless particle, $m_0 = 0$, so $pc = E$ or $p = E/c$, a result we derived by different means in Chapter 13, Eq. (13.13). For massless particles (notably photons), the norm of the four-momentum is 0 and the particle must move at the speed of light. The four-momentum for a massless particle has the form

$$\mathbf{P} = \frac{E}{c}(n_x, n_y, n_z, 1), \quad (14.26)$$

where $\hat{\mathbf{n}}$ is a unit vector in the direction of propagation.

For a particle of non-zero mass moving at low velocity

$$m = \gamma m_0 = \left(1 + \frac{1}{2} \frac{v^2}{c^2} + \dots\right).$$

Consequently

$$E = m_0c^2 + \frac{1}{2}m_0v^2 + \dots = m_0c^2 + K + \dots \quad (14.27)$$

where K is the Newtonian kinetic energy. In the low velocity approximation, the total energy of a free particle is the sum of its kinetic energy and its rest energy. Whether this interpretation is reasonable depends on the experimental evidence.

From Eq. (14.25), we have

$$E^2 - (pc)^2 = (m_0c^2)^2. \quad (14.28)$$

The rest energy of a particle is evidently the relativistic invariant of the four-momentum.

Postulating $mc = E/c$ led us directly to $E = mc^2$, the most famous formula in science. Of course, the validity of this formula does not depend on its fame but rather on the role energy plays in relativistic dynamics, a role that can be made meaningful only by experience.

In Chapter 13 we derived expressions for relativistic energy and momentum by applying the Lorentz transformation to observations of collisions in an isolated system. Those arguments rested on our understanding of collision processes and our intuition about the symmetry of views by different observers. In this chapter we found the same results by totally different reasoning—by examining the transformation properties of four-vectors. This argument is mathematically elegant and also physically elegant, for it reveals a deep connection between energy and momentum. Understanding phenomena from totally different lines of argument adds to our confidence in the truth of the explanation and cannot fail to deepen one's pleasure in physics.

14.7 Epilogue: General Relativity

Einstein was dissatisfied with his 1905 paper on special relativity because it could not deal with gravity. For example, according to Newton's

law of universal gravitation, changes in a gravitational field, due, for instance, to motion of the source mass, are felt instantaneously everywhere. According to special relativity, the effect could not propagate faster than the speed of light.

There is, however, a more profound dilemma. Newtonian gravity rests on the assumption that gravitational mass and inertial mass are identical. Einstein used this simple observation to motivate his new theory. He explained his line of reasoning with one of his famous “gedanken” (thought) experiments: the Einstein elevator. Because all things fall at the same rate, an observer in a stationary elevator in a gravitational field sees dropped objects accelerate downward, and would make the same observation if the elevator were accelerating upward in the absence of a gravitational field. Einstein pointed out that if the elevator were in space far from other bodies, then except for looking outside, there was no way to tell whether the elevator was accelerating up at rate g or at rest in a gravitational field g . He concluded that in a local region, there is no way to distinguish a downward gravitational field g from an upward acceleration of the coordinate system a . This observation is the principle of equivalence.

The principle of equivalence poses an immediate obstacle to the special theory of relativity. If a downward gravitational field is equivalent to an upward acceleration, then motion in a gravitational field is indistinguishable from motion in an accelerating coordinate system. Accelerating systems are inherently non-inertial. The crux of the dilemma is that the special theory of relativity is restricted to observations in inertial systems.

The special theory of relativity was published when Einstein was young and he presented it in a single paper that was comprehensible to all. In contrast, Einstein labored more than ten years to solve the problem of gravity, making a number of false starts. When he published his general theory of relativity in 1916 the paper was so complex that few readers could follow it. Furthermore, the only experimental confirmation was his theory’s ability to account for the discrepancy of 43 arc-second/century in the precession of the perihelion of Mercury that we mentioned in Section 10.6.

Such a result might seem like a minor detail of planetary dynamics. However, general relativity also made a dramatic prediction: light is deflected by gravity. In particular, the path of light from a star would be slightly bent as it passed near our massive Sun. The deflection would be so small that it could be observed only during a solar eclipse when the Sun’s brightness was blotted out for a short time, allowing stars in the part of the sky near the Sun to be seen. Observations were delayed by World War I but in 1919 two eclipse expeditions observed the effect. The theory of general relativity made the front page of newspapers, and the Einstein legend was born.

In 1919 cosmology was a topic for speculation but it was not part of science. Today we are in a golden age of cosmology and Einstein’s

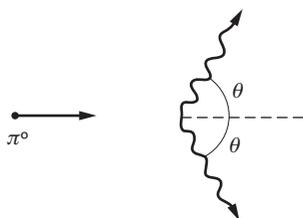
general theory of relativity is its foundational paradigm. The expansion of the universe is described by Einstein's field equations and the signature of his theory is to be found everywhere. For example, he predicted gravitational lensing—the focusing of light from a distance source that passes around a star or through a gravitating medium such as a galaxy. Today, gravitational lensing is a well-established tool in astrophysics.

Perhaps the most dramatic experimental prediction of general relativity is the existence of black holes. If the bending of light by gravity is large enough, the light cannot propagate. This occurs when the energy required to overcome the gravitational attraction of a mass, GMm/r , exceeds the rest mass mc^2 . Equating these gives the radius of the black hole as $R = GM/c^2$. The radius is typically only a few km, but the density is so great that the mass is huge. We saw in Example 10.9 that the mass of the black hole Sgr A* at the center of our galaxy is 4 million times the mass of the Sun. Not even light can escape from such an intense gravitational field—hence the name *black hole*. Fortunately, material falling inward radiates intensely and many black holes have been identified. Measured data for the orbit of a star around Sgr A* appear in Example 10.9.

Einstein launched two lines of progress in 1905: his photoelectric effect paper was seminal to the creation of quantum mechanics, and his paper on special relativity inspired general relativity. Quantum theory and gravitational theory are triumphs of twentieth century physics that changed our world view. These separate theories have yet to be reconciled: a quantum theory of gravity has yet to be formulated. Physics is never finished.

Problems

For problems marked *, refer to page 525 for a hint, clue, or answer.



14.1 *Pi meson decay**

A neutral pi meson (π^0), rest mass 135 MeV, decays symmetrically into two photons while moving at high speed. The energy of each photon in the laboratory system is 100 MeV.

(a) Find the meson's speed V as a ratio V/c .

(b) Find the angle θ in the laboratory system between the momentum of each photon and the initial line of motion.

14.2 *Threshold for pi meson production**

A high energy photon (γ -ray) collides with a proton at rest. A neutral pi meson (π^0) is produced according to the reaction $\gamma + p \rightarrow p + \pi^0$.

What is the minimum energy the γ -ray must have for this reaction to occur? The rest mass of a proton is 938 MeV, and the rest mass of a π^0 is 135 MeV.

14.3 *Threshold for pair production by a photon*

A high energy photon (γ -ray) collides with an electron and produces an electron–positron pair according to the reaction $\gamma + e^- \rightarrow e^- + (e^- + e^+)$.

What is the minimum energy the γ -ray must have for the reaction to occur?

14.4 *Particle decay*

A particle of rest mass M spontaneously decays from rest into two particles with rest masses m_1 and m_2 .

Show that the energies of the particles are $E_1 = (M^2 + m_1^2 - m_2^2)c^2/2M$ and $E_2 = (M^2 - m_1^2 + m_2^2)c^2/2M$.

14.5 *Threshold for nuclear reaction**

A nucleus of rest mass M_1 moving at high speed with kinetic energy K_1 collides with a nucleus of rest mass M_2 at rest. A nuclear reaction occurs according to the scheme $M_1 + M_2 \rightarrow M_3 + M_4$ where M_3 and M_4 are the rest masses of the product nuclei.

The rest masses are related by $(M_3 + M_4)c^2 = (M_1 + M_2)c^2 + Q$, where $Q > 0$. Find the minimum value of K_1 required to make the reaction occur, in terms of M_1 , M_2 , and Q .

14.6 *Photon-propelled rocket*

A rocket of initial mass M_0 starts from rest and propels itself forward along the x axis by emitting photons backward.

(a) Show that the four-momentum of the rocket's exhaust in the initial rest system can be written $\mathbf{P} = \gamma M_f v(-1, 0, 0, i)$, where M_f is the final mass of the rocket. (Note that this result is valid for the exhaust as a whole even though the photons are Doppler-shifted.)

(b) Show that the final velocity of the rocket relative to the initial frame is

$$v = \frac{\mu^2 - 1}{\mu^2 + 1}c,$$

where $\mu = M_0/M_f$ is the ratio of the rocket's initial mass to its final mass.

14.7 *Four-acceleration**

Construct a four-vector \mathbf{A} representing acceleration. For simplicity, consider only straight line motion along the x axis. Let the instantaneous four-velocity be $\mathbf{U} = \gamma(u, 0, 0, c)$.

14.8 *A wave in spacetime*

The function $f(x, t) = A \sin 2\pi[(x/\lambda) - \nu t]$ represents a sine wave of frequency ν and wavelength λ . The wave propagates along the x axis with velocity = wavelength \times frequency = $\lambda\nu$.

$f(x, t)$ can represent a light wave; A then corresponds to some component of the electromagnetic field that constitutes the light signal, and the wavelength and frequency satisfy $\lambda\nu = c$.

Consider the same wave in the coordinate system (x', y', z', t') moving along the x axis at velocity v . In this reference frame the wave has the form

$$f'(x', t') = A' \sin 2\pi \left(\frac{x'}{\lambda'} - \nu' t' \right).$$

(a) Show that the velocity of light is correctly given provided that $1/\lambda'$ and ν' are components of a four-vector \mathbf{K} given in the (x, y, z, t) system by

$$\mathbf{K} = 2\pi \left(\frac{1}{\lambda}, 0, 0, \frac{\nu}{c} \right).$$

(b) Using the result of part (a), derive the result for the longitudinal Doppler shift by evaluating the frequency in a moving system.

(c) Extend the analysis of part (b) to find the expression for the transverse Doppler shift, by considering a wave propagating along the y axis.