Planck’s constant $h = 6.626 \times 10^{-34}$ kg m$^2$/s, or $\hbar = h/2\pi = 1.055 \times 10^{-34}$ kg m$^2$/s

Time dependent Schrödinger equation: $i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t)$

General solution $\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_nt/\hbar}$

Time independent Schrödinger equation for states with energy $E$:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = E\psi$$

Infinite square well (width $a$) energies $E_n = \left(\frac{\pi^2}{2ma^2}\right) n^2$, where $n = 1, 2, 3,\ldots$

Lowest (ground) state wavefunctions $\psi_{n=0}(x)$ of the harmonic oscillator, with $x_0 = \sqrt{\frac{\hbar}{2m\omega}}$

$$\psi_0(x) = \left(\frac{1}{\sqrt{2\pi x_0}}\right)^{\frac{1}{2}} e^{-x^2/(4x_0^2)}$$

1. **Fun Quick Problems**

(a) [5] We derived the condition for the circular Bohr orbits of an electron around its nucleus that the circumference must be an integral multiple $n$ of the deBroglie wavelength $\lambda = h/p$. What is the equivalent condition on the values of angular momentum $L = r \times p$ of the electron?

For circular motion, $r$ is perpendicular to $p$, so $L = rp = \frac{n}{2\pi} \left(\frac{h}{p}\right) p = n \left(\frac{h}{2\pi}\right)$

(b) [5] What are the units of the wavefunction of a particle in 1 spatial dimension? Justify your answer.

$$\int_{-\infty}^{+\infty} |\varphi(x)|^2 dx = 1, \text{ so } \varphi(x) \propto \frac{1}{\sqrt{\text{length}}}$$

(c) [5] A particle has a pretty well-defined momentum, with uncertainty $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = 5.275 \times 10^{-29}$ kg m/s. What is the minimum uncertainty of the particle’s position $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$?

$$\Delta p \Delta x \leq \frac{\hbar}{2}, \text{ so } \Delta x \leq \frac{\hbar}{2\Delta p} = \frac{1.055 \times 10^{-34}}{2 \times 5.275 \times 10^{-29}} = 10^{-6} m$$
2. Quantum Mechanics in 1D potentials

(a) [5] A massive particle moves in 1D and is subjected to the potential energy function $V(x)$ plotted below. Mark the table with an “X” concerning states with energy $E$ indicated.

<table>
<thead>
<tr>
<th>Energy range</th>
<th>Any value OK</th>
<th>Quantized values only</th>
<th>No values allowed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E &lt; -V_0$</td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>$-V_0 &lt; E &lt; 0$</td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E &gt; 0$</td>
<td></td>
<td>X</td>
<td></td>
</tr>
</tbody>
</table>

(b) [10] For the same 1D potential energy function in (a), a particle has energy $E$ in the range $-V_0 < E < 0$. Sketch a possible wavefunction $\psi(x)$ for this stationary state in the graph provided. (Make sure to line up the $x$-axis with the potential function above!)
3. **Measuring bound particles**

(a) [10] A particle of mass \( m \) is in the ground state of a 1D harmonic oscillator with energy \( \frac{\hbar \omega}{2} \). Find the probability that an observation of the position \( x \) of this particle will detect the particle in the classically forbidden region where \( \frac{1}{2} m \omega^2 x^2 > E \).

**HINT:** \[ \int_0^1 e^{-z^2} dz \approx 0.75 \]

\[
P_{\text{forbidden}} = 1 - \frac{1}{\sqrt{\pi} \sigma} \int_{-\infty}^{\infty} e^{-x^2/4\sigma^2} dx = 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz = 1 - \frac{2}{\sqrt{\pi}} \left( \frac{1}{0.75} \right) = 0.154
\]

(b) [5] A particle of mass \( m \) is confined in an infinite square well of width \( a \) and prepared in a state of definite energy \( E = \left( \frac{\pi^2 \hbar^2}{2ma^2} \right) n^2 \) where \( n \) is a nonzero integer. Its position is then measured and found to be precisely in the middle of the well, a distance \( a/2 \) from each wall. What can you say about the possible values of \( n \) ?

\[ n \text{ must be odd } \text{ (for } n \text{ even, } \phi(\text{middle}) = 0) \]