Supplemental Material

In this section, we aim to prove the following identity by induction,

$$\int_0^1 dx \mathrm{W}(2^n-1,x)e^{i2^{n+1}\pi x}\sum_{l=0}^{n} a_l x^l = 0. \quad (1)$$

The construction of the Walsh functions is simple in terms of the elementary sequences known as Rademacher functions $\mathrm{R}\,(n,t) = \mathrm{sign}\,[\sin{(2^n \pi t)}]$. The dyadic ordering of the Walsh functions allow them to be defined in terms of the Rademacher functions as $\mathrm{W}\,(n,t) = \prod_{i=1}^{m+1}\mathrm{R}\,(i,t)^{b_{i-1}}$ when n is expressed as a binary number $n = b_m 2^m + ... + b_0 2^0$ and $b_i = 0$ or 1. With this definition, it is easy to see that choosing the index 2^n-1 for the Walsh function means that all the b_i coefficients are 1. We now prove the base case, $n=1$.

$$\begin{aligned}
&\int_0^1 dx \mathrm{W}(1,x)e^{i4\pi x}\sum_{l=0}^{1} a_l x^l \\
&= \int_0^1 dx \mathrm{R}(1,x)e^{i4\pi x}\sum_{l=0}^{1} a_l x^l \\
&= a_1 \int_0^1 dx \mathrm{R}(1,x)e^{i4\pi x} x \\
&= a_1 \left(\int_0^{1/2} dx e^{i4\pi x} x - \int_{1/2}^{1} dx e^{i4\pi x} x \right) \\
&= a_1 \int_0^{1/2} dx e^{i4\pi x}\left(x-(x+1/2)\right) = 0.
\end{aligned}$$

For the inductive step, assume that (4) is true and look at the $2^{n+1}-1$ case:

$$\begin{aligned}
&\int_0^1 dx \mathrm{W}(2^{n+1}-1,x)e^{i2^{n+2}\pi x}\sum_{l=0}^{n+1} a_l x^l \\
&= \int_0^1 dx \mathrm{R}(1,x)....\mathrm{R}(n+1,x)e^{i2^{n+2}\pi x}\sum_{l=0}^{n+1} a_l x^l \\
&= \frac{1}{2}\int_0^2 dx \prod_{i=1}^{n+1}\mathrm{R}(i,x/2)e^{i2^{n+1}\pi x}\sum_{l=0}^{n+1} a_l \left(\frac{x}{2}\right)^l \\
&= \frac{1}{2}\int_0^1 dx \prod_{i=1}^{n+1}\mathrm{R}(i,x/2)e^{i2^{n+1}\pi x}\sum_{l=0}^{n+1} a_l \left(\frac{x}{2}\right)^l \\
&+ \frac{1}{2}\int_1^2 dx \prod_{i=1}^{n+1}\mathrm{R}(i,x/2)e^{i2^{n+1}\pi x}\sum_{l=0}^{n+1} a_l \left(\frac{x}{2}\right)^l .
\end{aligned}$$

For the next step, note that for an integer $n \geq 0$, $\mathrm{R}(n+1,x/2) = \mathrm{R}(n,x)$, which allows the expression to be written as,

$$\begin{aligned}
&= \frac{1}{2}\int_0^1 dx \prod_{i=0}^{n}\mathrm{R}(i,x)e^{i2^{n+1}\pi x}\sum_{l=0}^{n+1} a_l \left(\frac{x}{2}\right)^l \\
&+ \frac{1}{2}\int_1^2 dx \prod_{i=0}^{n}\mathrm{R}(i,x)e^{i2^{n+1}\pi x}\sum_{l=0}^{n+1} a_l \left(\frac{x}{2}\right)^l \\
&= \frac{1}{2}\int_0^1 dx \prod_{i=1}^{n}\mathrm{R}(i,x)e^{i2^{n+1}\pi x}\sum_{l=0}^{n+1} a_l \left(\frac{x}{2}\right)^l \\
&- \frac{1}{2}\int_1^2 dx \prod_{i=1}^{n}\mathrm{R}(i,x)e^{i2^{n+1}\pi x}\sum_{l=0}^{n+1} a_l \left(\frac{x}{2}\right)^l .
\end{aligned}$$

In the next step, the substitution $x' = x-1$ is made and we take advantage of the fact that $\mathrm{R}(n,x+1) = \mathrm{R}(n,x)$ for $n \geq 1$.

$$\begin{aligned}
&= \frac{1}{2}\int_0^1 dx \mathrm{W}(2^n-1,x)e^{i2^{n+1}\pi x}\sum_{l=0}^{n+1} \frac{a_l}{2^l} x^l \\
&- \frac{1}{2}\int_0^1 dx \mathrm{W}(2^n-1,x)e^{i2^{n+1}\pi x}\sum_{l=0}^{n+1} \frac{a_l}{2^l}\sum_{k=0}^{l}\binom{l}{k} x^k \\
&= \frac{1}{2}\int_0^1 dx \mathrm{W}(2^n-1,x)e^{i2^{n+1}\pi x}\sum_{l=0}^{n} b_l x^l,
\end{aligned}$$

which is zero by assumption since b_l is a constant, thus concluding the proof. If the Radamacher functions are considered to be elementary sequences, then the Walsh filter is optimized in this resource for the task of suppressing the errors in the spin-dependent force operation that are discussed in this paper.