

Control of Trapped-Ion Quantum States with Optical Pulses

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We present new results on the quantum control of systems with infinitely large Hilbert spaces. A control-theoretic analysis of the control of trapped-ion quantum states via optical pulses is performed. We demonstrate how resonant bichromatic fields can be applied in two contrasting ways — one that makes the system completely uncontrollable and the other that makes the system controllable. In some interesting cases, the Hilbert space of the qubit-harmonic oscillator can be made finite, and the Schrödinger equation controllable via bichromatic resonant pulses. Extending this analysis to the quantum states of two ions, a new scheme for producing entangled qubits is discovered.

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Quantum computers rely on the quantum coherence exhibited by physical systems and on the understanding of their control. In this Letter, we apply theoretical concepts of quantum control to a scalable quantum-computing paradigm—a crystal of trapped ions [1]. The two-level atom (qubit) coupled to a harmonic oscillator is an example of a quantum system with an infinitely large number of accessible eigenstates. The control of such systems is not well understood. Recently, it was argued based on compactness arguments that infinite-dimensional systems are not controllable [2]. We provide a counterexample to this long-held view [3] — the bichromatic control of trapped-ion quantum states.

This work was motivated by the need to develop fast control schemes to produce entangled states of qubits. Such entangled states could then lead to interesting quantum states of the coupled spin-motion system. There has been much interest recently in applying techniques from coherent control to discover schemes to perform quantum algorithms and universal gate operations [4]. A recent paper shows a method of creating fast control-phase gates in trapped ions [5]. We perform a theoretical analysis of the control of trapped-ion quantum states with bichromatic fields. We show that in some interesting cases of resonant bichromatic control, the Hilbert space of the qubit-harmonic oscillator system can be made finite. In these cases, the Schrödinger equation is controllable. We show that the controllability can be extended to two-ion quantum states, thus providing a new scheme for qubit entanglement.

A trapped-ion qubit is most readily formed of two hyperfine states of a laser-coolable ion, separated by a frequency $\omega_0/2\pi$ in the several GHz range. Qubits are coupled via the vibrational modes of the ions' motion, which can be treated as quantum harmonic oscillators [1]. The quantized vibrational energy levels separated by a frequency $\omega_m/2\pi$ in the MHz range create sidebands in the spectrum of the ion. The hyperfine “qubit” states are addressed by a pair of optical beams (with counterpropagating wave vector components) in the Raman (lambda)

configuration, far detuned from an excited state that can be adiabatically eliminated. This interaction can be described by a control Hamiltonian for an ion interacting with an electromagnetic field $\mathbf{E}(\hat{\mathbf{r}}, \mathbf{t}) = \mathbf{E}(\mathbf{t})\hat{\mathbf{x}} \times \cos(\mathbf{kz} - \omega_L \mathbf{t})$ via an effective dipole moment $\hat{\boldsymbol{\mu}}$ [1], and is written (in atomic units, where $e = m_e = \hbar = 1$) as $H_I = -\hat{\boldsymbol{\mu}} \cdot \mathbf{E}(\hat{\mathbf{r}}, \mathbf{t})$. Choosing an interaction picture where we can rotate away the fast ω_0 contributions [$|\phi\rangle = \exp(i\frac{1}{2}\sigma_z\omega_0 t)|\psi\rangle$], and making the standard rotating wave approximation, the field-free (drift) Hamiltonian $H'_0 = \sum_m \omega_m \hat{n}_m$, and the control Hamiltonian, is written as

$$H'_I = \sum_i \Omega^i(t) [\sigma_+^i e^{i[\Delta t + \sum_m M_m^i \eta_m (a_m + a_m^\dagger)]} + \text{H.c.}] \quad (1)$$

The Pauli operator $\boldsymbol{\sigma}$ describes the equivalent spin- $\frac{1}{2}$ system represented by the qubit. The detuning of the field's central frequency ω_L from ω_0 , Δ , is usually zero. The ion's position operator \hat{z} is expanded in normal mode coordinates as $\hat{z} = z_{0m}(a_m + a_m^\dagger)$, z_{0m} is the spatial extent of the ground state wave function for the ion and mode being addressed, and M_m^i is the relation (a matrix) between the position of the i th ion and the normal mode coordinates. The Lamb-Dicke parameter $\eta_m = kz_{0m}$ is a coupling parameter between an ion qubit and a motional mode when radiation at wave vector k is applied. $\Omega^i(t) = \frac{\mu^i E(t)}{4}$ has the units of a Rabi frequency.

First, we present an analysis of the resonant control of the coupled harmonic oscillator-spin- $\frac{1}{2}$ system in the Lamb-Dicke limit (LDL), where the extent of the ions' motion is much smaller than the wavelength of the applied field. For compactness, the superscript i is dropped when discussing a single ion. We represent the various eigenstates $|\hat{S}, n_m\rangle$ by vertices of a graph as shown in Fig. 1. When a resonant electromagnetic field is applied, the coupling between two eigenstates caused by the interaction form the edges. This graph is similar to the “connectivity graph” of Ref. [2] and figures in Ref. [6], with two differences that aid in our understanding of the control mechanisms: the eigenstates are ordered in

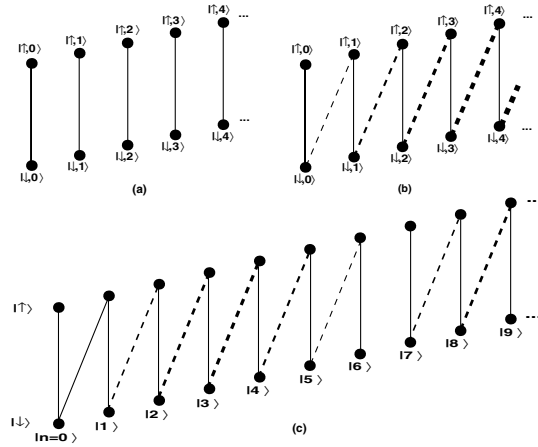


FIG. 1. (a) Lamb-Dicke limit (LDL), monochromatic field of frequency ω_c ; (b) LDL, bichromatic field with frequencies ω_c (solid edges) and ω_b (dashed edges); (c) beyond LDL, bichromatic field with frequencies ω_c and ω_b . The Lamb-Dicke parameter η has been chosen so that the $|\downarrow, 6\rangle$ to $|\uparrow, 7\rangle$ coupling vanishes.

energy, and the edges on the graph represent the matrix elements of the interaction between the eigenstates (not a population flow between them), their thickness qualitatively indicating the strength of the coupling.

For example, a field of frequency $\omega_L = \omega_c = \omega_0$ (the carrier frequency) acting on an ion connects states $|\downarrow n\rangle$ and $|\uparrow n\rangle$. Similarly, a field of frequency $\omega_L = \omega_b = \omega_0 + \omega_m$ (the first blue sideband) connects states $|\downarrow n\rangle$ and $|\uparrow n + 1\rangle$. In the LDL, it is well known that the strength of the coupling due to the carrier is independent of the phonon number of the eigenstates, whereas the couplings due to the blue sideband increases as the square root of the vibrational quantum number of the eigenstates.

We consider the simultaneous application of light of two frequencies ω_c and ω_b in the LDL. The graph in Fig. 1(b) indicates that this bichromatic field makes the graph transitively connected. That is, there exists a non-zero (direct or indirect) coupling between every pair of eigenstates. But the unbounded nature of the coupling operator leads to the infinite dimensionality of the accessible state space. The problem is exacerbated because the strength of the coupling due to the blue light increases as the square root of the vibrational quantum number n_m . These features make the system uncontrollable; i.e., an arbitrary superposition of eigenstates cannot be created. In fact, even a single eigenstate cannot be constructed. (For this same reason, an infinite-level quantum harmonic oscillator is not controllable with a resonant dipole field.) This can be rigorously demonstrated by calculating the Lie algebra formed by the time-independent parts of Hamiltonians H'_0 and H'_1 , which does not close [7], and whose span increases with successive elements of the algebra, thus making it impossible to access only the desired state.

In many cases (such as Rydberg atoms or molecules), an infinite-dimensional system can be numerically ap-

proximated by truncating the state space to an essential (finite) basis. This is possible because the strength of the couplings due to the applied field drops off outside this essential state space, and, to a good approximation, the coupling operator is bounded. In the coupled spin- $\frac{1}{2}$ -harmonic oscillator system, the situation is exactly the opposite because of the infinitely many degenerate transitions. Thinking in terms of control via interfering paths [8], there are an infinite number of likely paths that connect an initial state to a final state. Therefore, truncating the state space cannot approximate this system. This analysis indicates that there is no field made of the carrier and blue frequencies that can completely control the coupled harmonic oscillator-spin- $\frac{1}{2}$ system. The analysis is identical for bichromatic fields with the carrier and red frequencies.

Contrary to this analysis, in 1996, Law and Eberly [9] showed that by using the carrier and red fields *alternately* it is possible to create arbitrary finite superpositions of harmonic oscillator states. Reference [10] shows that the same scheme can be used to generate any finite superposition of states in a spin- $\frac{1}{2}$ /harmonic oscillator system. This type of control was also the basis of the quantum-computing scheme prescribed by Cirac and Zoller [11]. We now analyze this scheme from the control-theoretic viewpoint.

As seen in the graph in Fig. 1(a), a monochromatic field reduces the state space of the infinite-dimensional problem to (infinite copies of) the two-level problem, and the resonant control of a two-level system is well understood. Indeed, using two colors ω_0 and ω_b alternately, any eigenstate can be reached from any other eigenstate. We can also interpret the physics of this system by looking at the infinitesimal propagator due to the bichromatic field, which is expanded using the Baker-Campbell-Hausdorff formula.

$$e^{i(H_c + H_b)\delta t} = e^{iH_c\delta t} e^{iH_b\delta t} e^{-\frac{i}{2}[H_c, H_b]\delta t^2} \times e^{-\frac{i}{12}([H_c, [H_c, H_b]] - [H_b, [H_c, H_b]])\delta t^3} \dots \quad (2)$$

The higher the order of the commutator, the greater is the span of the space produced by the propagator acting on an eigenstate. Because the coupling strength of the blue transition increases with an increase in n , this series never truly truncates. However, if either Ω_c or Ω_b is turned off, the series collapses beautifully to only the first or second term. In this way, a theoretically uncontrollable system can be made completely controllable. We note that, in practice, the two colors should be turned on adiabatically so that a single color field acts on the system at any time. This is a result contrary to the accepted notion of control that the controllability of quantum systems does not critically depend on the specific temporal profile of the control field.

Thus, the Law-Eberly scheme [9] is the only explicit scheme for accessing finite superpositions of trapped-ion quantum states in the Lamb-Dicke limit. However, the

method requires sequential monochromatic pulses, each turned on and off adiabatically. Therefore, it takes several trap periods to complete a logic gate operation. This motivated our investigation of the control of the coupled spin- $\frac{1}{2}$ /harmonic oscillator system via shorter optical pulses with multiple resonant colors.

In some interesting circumstances (described below), it is possible to achieve bichromatic control over the system. Consider a case when the Lamb-Dicke criterion is not satisfied. By setting $\omega_L = \omega_0$ in Eq. (1), a matrix element of the interaction Hamiltonian in the field-free eigenbasis can be written as

$$\langle S'n'|H_I|Sn\rangle = \Omega(t)2\text{Re}\{\langle S'|\sigma_+|S\rangle \otimes \langle n'|\exp[i(M_m^i\eta_m(a_m + a_m^\dagger))]|n\rangle\}, \quad (3)$$

where $M_m^i = 1$ for one ion. The harmonic oscillator part of this matrix element [1] is written as

$$\langle n'|\exp[i(\eta_m(a_m + a_m^\dagger))]|n\rangle = \exp(-\eta_m/2)\sqrt{\frac{n_{<}}{n_{>!}}}\eta_m^{|n'-n|}L_{n_{<}}^{|n'-n|}(\eta_m^2). \quad (4)$$

The symbol $n_{>}$ refers to the larger of n and n' , and $n_{<}$ refers to the smaller of n and n' . $L_n^\alpha(x)$ is the associated Laguerre polynomial. When the applied field contains the frequencies ω_c and ω_b , the harmonic oscillator parts of the carrier and blue transition matrix elements have oscillatory behaviors as shown in Fig. 2. If the ion-trap is adjusted (thereby adjusting η_m) so that the coupling strength of one of the (blue or carrier) transitions becomes zero, the system is transformed into a finite closed subsystem and a remaining infinite subsystem. This is closely related to the technique of truncating classical polynomials used to calculate numerical and analytical solutions for the coherent dynamics of multilevel atoms and molecules [12]. In Fig. 1(c), the argument of the Laguerre polynomial η_m^2 is adjusted to 0.527 667 so that $L_6^1(\eta_m^2) = 0$, and the $|\downarrow, 6\rangle$ to $|\uparrow, 7\rangle$ transition is turned off. Similarly, the $|\downarrow, 5\rangle$ to $|\uparrow, 5\rangle$ transition can be turned off by adjusting $\eta_m^2 = 0.322 548$ so that $L_5^0(\eta_m^2) = 0$. Experimentally, the Lamb-Dicke parameter can be manipulated by adjusting the trap strength as shown by the recent implementation of the wave packet controlled-NOT (CNOT) gate [13].

We now examine this finite harmonic oscillator/spin- $\frac{1}{2}$ system for controllability. When the carrier and blue colors are applied, it is seen in Fig. 1(c) that the graph of this finite subsystem is transitively connected. Within the completely connected subspace, the fields cause degenerate transitions. However, the coupling strengths of the degenerate transitions are unequal. Therefore, the

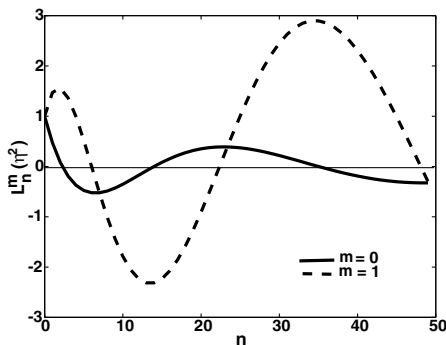


FIG. 2. Associated Laguerre polynomial for the $|n\rangle \rightarrow |n \pm m\rangle$ matrix elements, where $m = 0$ for the carrier transition and $m = 1$ for the blue sideband transition.

conditions for controllability of finite-dimensional systems [2,7] are satisfied. This implies that there exists a target time $t_f < \infty$, when any desired unitary transformation $U(0, t_f)$ can be performed within this truncated Hilbert space using a bichromatic field. Thus, by using very few parameters (the two field strengths, the relative phase between the two fields, and the target time), an arbitrary superposition of eigenstates within the finite Hilbert space can be created.

Note that this bichromatic control is still a relatively slow process. However, there is no requirement of multiple adiabatic turn-ons and turn-offs of the control fields, as in the sequential, monochromatic control scheme. It is expected that the precision of the pulse timing in this scheme would have to be much better than the time scale set by Ω^{-1} , but this is also true for conventional sequential schemes. Also, the termination of these resonant transitions and the faithful quantum control of the system requires that the time scale of the control pulses be much faster than the motional heating rate, which can cause the motional quantum state to vacate the closed subsystem. Motional heating in ion traps is not a fundamental limitation—recent experiments [14] have observed motional heating at much slower time scales than the control pulses proposed here.

We now show how this bichromatic control scheme can be used to produce entangled states of two qubits. For two trapped ions, there are two axial modes of vibration—the center of mass mode, and the breathing mode. Isolating one of the modes of vibration, we consider the problem of producing an entangled state of spin alone; that is, the vibrational contributions factor out.

Figure 3 shows the graph of the two-ion single-mode eigenstates. To aid visibility, the hyperfine splittings ω_0^i of the two ions are shown to be different. For two ions of the same mass, the magnitude of vibration is the same for both ions; therefore, $(M_m^i\eta_m)^2$ is the same. Therefore, the trap can be adjusted so that the spin- $\frac{1}{2}$ harmonic oscillator ladders truncate at exactly the same transition for both ions. (This truncation method does not work for more than two ions, because then the magnitudes of vibration of all ions are not equal.) When the two ions are individually addressed with bichromatic fields, the relevant colors are $\omega_b^1, \omega_c^1, \omega_b^2,$ and ω_c^2 . As seen in Fig. 3, the graph of the

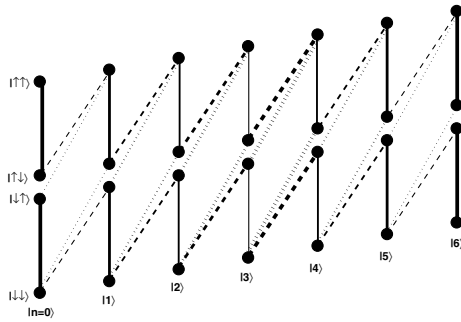


FIG. 3. Finite subspace of two ions. Three color resonant field with ω_b^1 (dashed line), ω_b^2 (dotted line), and ω_c^1 (solid line). η is adjusted to create a finite subspace.

finite subspace is transitively connected when fields ω_b^1 , ω_b^2 , and ω_c^1 are applied. This graph satisfies all the conditions required for the complete controllability of a finite system [2,7]. If the hyperfine splittings of the two ions are the same, $\omega_{b,c}^1$ is equal to $\omega_{b,c}^2$. This is a problem for controllability of the finite system, which can be resolved by individually addressing the two ions. This makes the degenerate transitions distinguishable, and the two-ion single-mode system completely controllable. Thus, by individually addressing two ions with resonant, bichromatic fields, it is possible to create an arbitrary final state from any initially pure quantum state of spin + motional finite subsystem. In particular, entangled states of two-ion qubits can be created where the motion is factored from the overall quantum state. For a scalable ion-trap quantum computer [15], it is sufficient to generate pairwise entanglement of two-ion qubits.

In contrast to the case of individual addressing, for multicolor, resonant control with uniform illumination of the ions, it is necessary to have ions with different ω_0 for controllability. This can be achieved experimentally by applying a magnetic field gradient along the trap axis. The gradient must be judiciously chosen to avoid resonances caused by a coupling between the two ions, or the rf field in the frame of the moving ions, or the interference of the other motional mode levels. More practically, the gradient scheme sets a speed limit roughly given by the differential Zeeman shift of the two ions. From our controllability analysis, we know that entangled spin states can be created using resonant, multicolor pulses. However, this feature of having different ω_0 makes it impractical to calculate pulse solutions. The interaction Hamiltonian is similar to Eq. (1), with a summation over the terms for each ion. If the ω_0^i s are different, the detunings $\omega_0^i - \omega_L$ cannot all be set to zero. The time-dependent term in Eq. (1) makes it necessary to diagonalize the interaction matrix at every time step. This increases the time to perform such calculations beyond reasonable limits. Therefore, we propose an experiment where the three-color pulse (with colors ω_b^1 , ω_b^2 , and ω_c^1) that can produce entangled spin states of the trapped ions is generated via learning control [16]. The amplitudes and

phases of the constituent colors form a small parameter space to be searched. It will be expedient to experimentally implement the control of trapped ions via three-color pulse shaping and allow the ions themselves to “solve” the time-dependent Schrödinger equation. This scheme is an example of a quantum simulation of a calculation that is intractable on a classical computer [17].

In summary, we have demonstrated that infinite-dimensional quantum systems can be made completely controllable in certain circumstances. Our results show that the specific temporal shapes of control fields are important in establishing controllability of quantum systems. A bichromatic control scheme that leads to the finite-dimensionality and controllability of the trapped-ion system is presented. This scheme can be extended to produce entangled states of two trapped-ion qubits.

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