QM background

1. States

The quantum state (wavefunction, etc.) is a complete description of a quantum system and is described by a state vector

\[ |\psi\rangle \]

in Hilbert space \( \mathcal{H} \)

Hilbert space:

- a vector space over complex numbers \( \mathbb{C} \)
- has an inner product (norm) that maps an ordered pair of vectors in \( \mathcal{H} \) into \( \mathbb{C} \)

\[
\langle \phi | \psi \rangle = a + bi
\]

\[
\langle \phi | \phi \rangle = 1
\]

\( |\phi\rangle \) and \( e^{i\theta} |\phi\rangle \) describe same state

\( |\phi\rangle \) and \( |\psi\rangle \) are in \( \mathcal{H} \), then so is \( \alpha |\phi\rangle + \beta |\psi\rangle \)

2. Observables – properties of a system which can be measured.

All observables are operators in \( \mathcal{H} \) and have real eigenvalues (self-adjoint or Hermitian).

\[
\hat{M} = \hat{M}^* \quad [\hat{M}^* = (M^*)^{\dagger}]
\]

Observable operators map state vectors from place to place in \( \mathcal{H} \). (They are matrices!)

\[
\hat{M}: \quad |\phi\rangle \rightarrow \hat{M} |\phi\rangle
\]

\( \alpha |\phi\rangle + \beta |\psi\rangle \rightarrow \alpha \hat{M} |\phi\rangle + \beta \hat{M} |\psi\rangle \)
Eigenstates of Hermitian operators form orthonormal bases in \( \mathcal{H} \):

\[
\hat{M} = \sum_j r_j \hat{P}_j \quad \text{a diagonal matrix}
\]

where \( \hat{P}_j \) is a "projector" of eigenstate \( |n\rangle \) (no degeneracies !)

(3) Probabilistic Measurement of a quantum system

The outcome of a measurement of observable \( M \) is always an eigenvalue of \( \hat{M} \). Right after measurement, the state is the eigenstate of \( \hat{M} \) with the measured eigenvalue.

\[
|n\rangle \rightarrow \frac{\hat{P}_n |n\rangle}{\sqrt{\langle n|\hat{P}_n|n\rangle}}
\]

When measurement results in \( r_n \) and probability of measuring \( r_n \) is given by

\[
|P_n| = \langle n|\hat{P}_n|n\rangle
\]

\[
\langle n| = \begin{pmatrix} \alpha_0 \rangle \rangle \\ \langle \beta_1 \rangle \rangle \end{pmatrix}
\]

\[
\hat{M} = 10\langle 0|0\rangle \rightarrow \text{projector of } |0\rangle \text{. Observable tells if we are in state } |0\rangle
\]

\[
|P_0| = |\alpha_0^* + \beta_1^*\rangle \rangle 10\langle 0|0\rangle [10^* + \beta_1^*] = |\alpha_0^*|^2
\]

Individually if we measure \( \langle n|\hat{M}|n\rangle = 1 \) then

\[
|n\rangle \rightarrow |0\rangle
\]

if we measure \( \langle n|\hat{M}|n\rangle = 0 \) then

\[
|n\rangle \rightarrow |1\rangle
\]
4) Time Evolution of state

\[ i \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \]

Hamiltonian operator

\[ |\psi(t)\rangle = e^{-i\hat{H}t} |\psi(0)\rangle \quad \text{for } \hat{H} \text{ time-independent} \]

The 2LS or qubit

classical bits 0 or 1

\[ \hat{H} \sim q^2 \quad (2\text{-dimensional Hilbert space}) \]

\[ |\psi\rangle = a |0\rangle + b |1\rangle \]

\[ e^{-|\text{spin} \rangle \langle \text{spin}|} \text{ (spin-1/2 object) in a magnetic field} \]

\[ \hat{H} = -\hat{n} \cdot \vec{B} = \frac{e}{mc} \hat{n} \cdot \vec{B} \]

Review of Angular Momentum in QM

\[ \hat{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z) \]

\[ [\hat{S}_x, \hat{S}_y] = 0 \quad [\hat{S}_x, \hat{S}_z] = i\hbar \hat{S}_z \]

\[ [\hat{S}_y, \hat{S}_z] = 0 \quad [\hat{S}_y, \hat{S}_x] = i\hbar \hat{S}_x \]

\[ [\hat{S}_z, \hat{S}_x] = 0 \quad [\hat{S}_z, \hat{S}_y] = i\hbar \hat{S}_y \]

Write down common eigenstates of \( \hat{S}^2 \) and \( \hat{S}_z \)

\[ |S, m_s\rangle \]

\[ \hat{S}^2 |S, m_s\rangle = \hbar^2 s(s+1) |S, m_s\rangle \]

\[ \hat{S}_z |S, m_s\rangle = \hbar m_s |S, m_s\rangle \quad 1m_s \leq S \]

\[ \hat{S}_\pm |S, m_s\rangle = \frac{\hbar}{2} \sqrt{s(s+1) - m_s(m_s \pm 1)} |S, m_s \pm 1\rangle \]

where \( \hat{S}_\pm = \hat{S}_x \pm i \hat{S}_y \)